Technical Appendix to Accompany

"Increased (Platform) Competition Reduces (Seller) Competition"

by Shana Cui

Part I of this Technical Appendix provides detailed proofs of the formal conclusions in the paper. Part II of this Technical Appendix provides the numerical solutions for settings where $\theta_j = 1$ and where $\theta_j = 1.1$. Part III of this Technical Appendix considers the case where $\widetilde{\text{P}}$ is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$).

Equations, Conditions, and Assumptions from the Text

$$q_{kj}^P = \theta_j \, \alpha_j - \beta_j^P \, p_{kj}^P + \eta_j \, p_{kj}^S. \tag{1}$$

$$q_{kj}^S = \alpha_j - \beta_j^S \, p_{kj}^S + \eta_j \, p_{kj}^P. \tag{2}$$

$$\theta_j \, \alpha_j - \beta_j^P \, p_{kj}^P + \eta_j \, p_{kj}^S = 0 \;\; \Leftrightarrow \;\; p_{kj}^P = \frac{\theta_j \, \alpha_j + \eta_j \, p_{kj}^S}{\beta_j^P}. \tag{3}$$

$$q_{kj}^S = \alpha_j - \beta_j^S \, p_{kj}^S + \eta_j \, \frac{\theta_j \, \alpha_j + \eta_j \, p_{kj}^S}{\beta_j^P} = \alpha_j \left[1 + \frac{\theta_j \, \eta_j}{\beta_j^P} \right] - p_{kj}^S \left[\beta_j^S - \frac{(\eta_j)^2}{\beta_j^P} \right]. \tag{4}$$

$$q_{kj}^S = A_j - b_j^S \, p_{kj}^S. \tag{5}$$

$$A_j = \alpha_j \left[1 + \frac{\theta_j \, \eta_j}{\beta_j^P} \right] > 0 \;\; \text{and} \;\; b_j^S = \beta_j^S - \frac{[\eta_j]^2}{\beta_j^P} > 0. \tag{6}$$

$$A_j = \alpha_j \left[\frac{\beta_j^P + \theta_j \, \eta_j}{\beta_j^P} \right] \;\; \Leftrightarrow \;\; \alpha_j \left[\beta_j^P + \theta_j \, \eta_j \right] = \beta_j^P \, A_j. \tag{7}$$

$$b_j^S = \frac{\beta_j^S \, \beta_j^P - [\eta_j]^2}{\beta_j^P} \;\; \Leftrightarrow \;\; \beta_j^S \, \beta_j^P - [\eta_j]^2 = \beta_j^P \, b_j^S. \tag{8}$$

$$\beta_j^S \, \beta_j^P - [\eta_j]^2 > 0. \tag{9}$$

$$b_j^S = \beta_j^S \left[1 - \frac{(\eta_j)^2}{\beta_j^S \, \beta_j^P} \right] = \beta_j^S \left[1 - \Omega_j \right]. \tag{10}$$

$$Q_{kj}^S = B_k \left[1 - f_i \right] q_{kj}^S. \tag{11}$$

$$\widetilde{\Delta}_{kj} = \Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj}. \tag{12}$$

$$\Phi_{1j} \equiv 2 \, \beta_j^P \, \beta_j^S + \left[\eta_j \right]^2. \tag{13}$$

$$\Phi_{2j} \equiv 4 \, \beta_j^P \, \beta_j^S - \left[\eta_j \right]^2. \tag{14}$$

$$\Omega_j \equiv \frac{\left[\eta_j\right]^2}{\beta_j^P \, \beta_j^S}. \tag{15}$$

$$M_{kj} \equiv \frac{1}{2\,[\,1-\Omega_j\,]} \left\{ \frac{\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P} + \frac{\Omega_j\,[\,2+\Omega_j\,]\,\left[\,26-\Omega_j+2\,(\Omega_j)^2\,\right][\Delta_{kj}]^2}{2\,\beta_j^S\,[\,8+\Omega_j\,]^2} + \frac{\Omega_j\left[\,6+2\,\Omega_j+(\Omega_j)^2\,\right]\overline{\Delta}_{kj}\,\Delta_{kj}}{\eta_j\,[\,8+\Omega_j\,]} \right\}. \tag{16}$$

$$\phi_{kj} \equiv \left[\left(\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}\right)\frac{\widetilde{\Delta}_{ij}}{\Delta_{kj}}\right]^2. \tag{17}$$

$$\xi_1\left(\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right) \equiv \left[\frac{5\,\eta_j\,(\,2+\Omega_j\,)}{\beta_j^S\,(\,8+\Omega_j\,)}+\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]\left[\frac{\eta_j\,(\,2+\Omega_j\,)}{\beta_j^S\,(\,8+\Omega_j\,)}+\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]+\frac{4\,\beta_j^P\,[\,2+\Omega_j\,]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}. \tag{18}$$

$$\xi_2\left(\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right) \equiv \left[\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)}\right]^2\left[1+\frac{\eta_j}{\beta_j^P}\,\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]^2. \tag{19}$$

$$\overline{a} \equiv \beta_j^S\,[\,1-\Omega_j\,]\,,\; \overline{b} \equiv -\frac{4\,\eta_j\,[\,1-\Omega_j\,]}{8+\Omega_j}, \text{ and}$$

$$\overline{c} \equiv \frac{\beta_j^P\left[\,(\,5\,\Omega_j+4\,)\,(\,2+\Omega_j\,)^2-(\,8+\Omega_j\,)^2\,\right]}{[\,8+\Omega_j\,]^2}. \tag{20}$$

Condition FS $\Theta_k\,M_{k2} - \frac{\Theta_k\left[\widetilde{\Delta}_{k1}\right]^2}{8\,b_1^S} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\,b_2^S} < F < \min\{\,\Theta_k\,M_{k2} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\,b_2^S},\, \Theta_k\,M_{k1} - \frac{\Theta_k\left[\widetilde{\Delta}_{k1}\right]^2}{8\,b_1^S} - \frac{\Theta_k\left[\widetilde{\Delta}_{k2}\right]^2}{8\,b_2^S}\}$.

Assumption BC if $c_{1j}^P < c_{2j}^P$, then $\phi_1 > \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2$ for $j \in \{1,2\}$.

I. Proofs of Formal Conclusions.

Lemma 1. *Suppose Sj faces no competition from Pk when Sj sells on Pk* ($j,k \in \{1,2\}$). *Given* w_{kj}, *Sj's equilibrium output (i.e., sales)* (Q_{kj}^S) *is* $\frac{\Theta_k\left[\widetilde{\Delta}_{kj}-b_j^S\,w_{kj}\right]}{2}$, *and Sj's total profit is* $\frac{\Theta_k}{b_j^S}\left[\,q_{kj}^S\,\right]^2$ *where* $q_{kj}^S = \frac{Q_{kj}^S}{\Theta_k}$.

<u>Proof</u>**.** Suppose Sj faces no competition from Pk when Sj sells on Pk ($j,k \in \{1,2\}$). (11) implies that Sj's profit is given by ($j,k,i \in \{1,2\}$, $k \neq i$):

$$\pi_j = \left[\,p_{kj}^S - w_{kj} - c_j^S\,\right]\Theta_k\,q_{kj}^S. \tag{21}$$

(5) and (21) imply Sj chooses p_j^S to

$$\text{Maximize} \quad \left[p^S_{kj} - w_{kj} - c^S_j \right] \Theta_k \left[A_j - b^S_j \, p^S_{kj} \right] \quad \Rightarrow \quad \frac{\partial \pi_j}{\partial p^S_{kj}} \; = \; 0$$

$$\Leftrightarrow \quad A_j - b^S_j \, p^S_{kj} - b^S_j \left[p^S_{kj} - w_{kj} - c^S_j \right] \; = \; 0 \quad \Leftrightarrow \quad p^S_{kj}\left(w_{kj} \right) \; = \; \frac{A_j + b^S_j \, c^S_j + b^S_j \, w_{kj}}{2 \, b^S_j}. \tag{22}$$

(5) and (22) imply that consumers' initial demand for Sj's product is:

$$q^S_{kj} \; = \; A_j - b^S_j \, \frac{A_j + b^S_j \, c^S_j + b^S_j \, w_{kj}}{2 \, b^S_j} \; = \; \frac{A_j - b^S_j \, c^S_j - b^S_j \, w_{kj}}{2} \; = \; \frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2}. \tag{23}$$

(11) and (23) imply that Sj's sales are:

$$Q^S_{kj} \; = \; \Theta_k \, q^S_{kj} \; = \; \frac{\Theta_k \left[\widetilde{\Delta}_{kj} - b^S_j \, w_{kj} \right]}{2}. \tag{24}$$

(21), (22), and (24) imply that Sj's profit is:

$$\begin{aligned} \pi_j \; &= \; \left[\frac{A_j + b^S_j \, c^S_j + b^S_j \, w_{kj}}{2 \, b^S_j} - w_{kj} - c^S_j \right] \frac{\Theta_k \left[\widetilde{\Delta}_{kj} - b^S_j \, w_{kj} \right]}{2} \\ &= \; \Theta_k \left[\frac{A_j - b^S_j \, c^S_j - b^S_j \, w_{kj}}{2 \, b^S_j} \right] \frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2} \\ &= \; \Theta_k \left[\frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2 \, b^S_j} \right] \frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2} \; = \; \frac{\Theta_k}{b^S_j} \left[\frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2} \right]^2 . \quad \blacksquare \end{aligned} \tag{25}$$

Lemma 2. *Suppose Sj faces no competition from Pk when Sj sells on Pk ($j, k \in \{1, 2\}$). Then Pk's profit-maximizing commission for Sj is $w_{kj} = \frac{\widetilde{\Delta}_{kj}}{2 \, b^S_j}$.*

Proof. Suppose Sj faces no competition from Pk when Sj sells on Pk ($j, k \in \{1, 2\}$). (23) implies that Pk chooses w_{kj} to

$$\text{Maximize} \;\; \Pi_k \; = \; w_{kj} \, \Theta_k \, q^S_{kj} + \overline{\Pi}_{kl} \; = \; w_{kj} \, \Theta_k \, \frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2} + \overline{\Pi}_{kl} \tag{26}$$

$$\Rightarrow \quad \frac{\partial \Pi_k}{\partial w_{kj}} \; = \; 0 \quad \Leftrightarrow \quad \frac{\widetilde{\Delta}_{kj} - b^S_j \, w_{kj}}{2} - \frac{b^S_j \, w_{kj}}{2} \; = \; 0 \quad \Leftrightarrow \quad w_{kj} \; = \; \frac{\widetilde{\Delta}_{kj}}{2 \, b^S_j}, \tag{27}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from Sl ($j, k, l \in \{1, 2\}$, $j \neq l$), $\overline{\Pi}_{kl} > 0$ if Sl sells on Pk, and $\overline{\Pi}_{kl} = 0$ if Sl does not on Pk.[1] $\blacksquare$

[1] $\overline{\Pi}_{kl}$ does not include w_{kj}.

Lemma 3. *Suppose Sj faces no competition from Pk when Sj sells on Pk ($j, k \in \{1,2\}$). Then Sj's equilibrium output (Q^S_{kj}) is $\frac{\Theta_k \widetilde{\Delta}_{kj}}{4}$, Sj's profit is $\frac{\Theta_k [\widetilde{\Delta}_{kj}]^2}{16 b^S_j}$, and Pk's profit from the commission it collects from Sj is $\frac{\Theta_k [\widetilde{\Delta}_{kj}]^2}{8 b^S_j}$.*

Proof. Suppose Sj faces no competition from Pk when Sj sells on Pk ($j, k \in \{1,2\}$). Lemmas 1 and 2 imply that consumers' initial demand for product j is

$$q^S_{kj} = \frac{\widetilde{\Delta}_{kj} - b^S_j w_{kj}}{2} = \frac{\widetilde{\Delta}_{kj} - b^S_j \frac{\widetilde{\Delta}_{kj}}{2 b^S_j}}{2} = \frac{\widetilde{\Delta}_{kj} - \frac{\widetilde{\Delta}_{kj}}{2}}{2} = \frac{\widetilde{\Delta}_{kj}}{4}. \tag{28}$$

(11) and (28) imply that Sj's sales are:

$$Q^S_{kj} = \Theta_k q^S_{kj} = \frac{\Theta_k \widetilde{\Delta}_{kj}}{4}.$$

Lemma 1 and (28) imply that Sj's profit is

$$\pi_j = \frac{\Theta_k}{b^S_j} \left[q^S_{kj} \right]^2 = \frac{\Theta_k}{b^S_j} \left[\frac{\widetilde{\Delta}_{kj}}{4} \right]^2 = \frac{\Theta_k \left[\widetilde{\Delta}_{kj} \right]^2}{16 b^S_j}. \tag{29}$$

Lemma 2 and (28) imply that Pk's profit from charging a commission from Sj is.

$$\Pi_k = w_{kj} \Theta_k q^S_{kj} = \Theta_k \frac{\widetilde{\Delta}_{kj}}{2 b^S_j} \frac{\widetilde{\Delta}_{kj}}{4} = \frac{\Theta_k \left[\widetilde{\Delta}_{kj} \right]^2}{8 b^S_j}. \ \blacksquare \tag{30}$$

Lemma 4. *Suppose Sj competes against Pk when Sj sells on Pk ($j, k \in \{1,2\}$). Given w_{kj}, Sj's equilibrium output (Q^S_j) is $\frac{\Theta_k \left[\frac{\eta_j}{\beta^P_j} \overline{\Delta}_{kj} + 2 \Delta_{kj} - 2 \beta^S_j (1 - \Omega_j) w_{kj} \right]}{4 - \Omega_j}$, Pk's equilibrium output (Q^P_{kj}) is $\frac{\Theta_k \left[2 \overline{\Delta}_{kj} + \frac{\eta_j}{\beta^S_j} \Delta_{kj} - \eta_j (1 - \Omega_j) w_{kj} \right]}{4 - \Omega_j}$, and Sj's total profit is $\frac{\Theta_k}{\beta^S_j} \left[q^S_{kj} \right]^2$ where $q^S_{kj} = \frac{Q^S_{kj}}{\Theta_k}$.*

Proof. Suppose Sj competes against Pk when Sj sells on Pk ($j, k \in \{1,2\}$). (1) and (2) imply that Pk's profit is

$$\Pi_k = \left[p^P_{kj} - c^P_{kj} \right] \Theta_k q^P_{kj} - F + w_{kj} \Theta_k q^S_{kj} + \overline{\Pi}_{kl} \tag{31}$$

$$= \left[p^P_{kj} - c^P_{kj} \right] \Theta_k \left[\theta_j \alpha_j - \beta^P_j p^P_{kj} + \eta_j p^S_{kj} \right] - F + w_{kj} \Theta_k \left[\alpha_j - \beta^S_j p^S_{kj} + \eta_j p^P_{kj} \right] + \overline{\Pi}_{kl}; \tag{32}$$

Sj's profit is

$$\pi_j = \left[p^S_{kj} - w_{kj} - c^S_j \right] \Theta_k q^S_{kj} = \left[p^S_{kj} - w_{kj} - c^S_j \right] \Theta_k \left[\alpha_j - \beta^S_j p^S_{kj} + \eta_j p^P_{kj} \right], \tag{33}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from Sl ($j, k, l \in \{1, 2\}$, $j \neq l$), $\overline{\Pi}_{kl} > 0$ if Sl sells on Pk, and $\overline{\Pi}_{kl} = 0$ if Sl does not on Pk.

(32) implies that Pk chooses its price p^P_{kj} to

$$\text{Maximize } \; \Pi_k = \left[p^P_{kj} - c^P_{kj} \right] \Theta_k \left[\theta_j \, \alpha_j - \beta^P_j \, p^P_{kj} + \eta_j \, p^S_{kj} \right] - F + w_{kj} \, \Theta_k \left[\alpha_j - \beta^S_j \, p^S_{kj} + \eta_j \, p^P_{kj} \right] + \overline{\Pi}_{kl} \tag{34}$$

$$\Rightarrow \; \frac{\partial \Pi_k}{\partial p^P_{kj}} = 0 \; \Leftrightarrow \; \theta_j \, \alpha_j - \beta^P_j \, p^P_{kj} + \eta_j \, p^S_{kj} - \beta^P_j \left[p^P_{kj} - c^P_{kj} \right] + w_{kj} \, \eta_j = 0$$

$$\Leftrightarrow \; p^P_{kj} = \frac{\theta_j \, \alpha_j + \eta_j \, p^S_{kj} + \beta^P_j \, c^P_{kj} + w_{kj} \, \eta_j}{2 \, \beta^P_j}. \tag{35}$$

(33) implies that Sj chooses its price p^S_{kj} to

$$\text{Maximize } \; \pi_j = \left[p^S_{kj} - w_{kj} - c^S_j \right] \Theta_k \left[\alpha_j - \beta^S_j \, p^S_{kj} + \eta_j \, p^P_{kj} \right] \; \Rightarrow \; \frac{\partial \pi_j}{\partial p^S_{kj}} = 0$$

$$\Leftrightarrow \; \alpha_j - \beta^S_j \, p^S_{kj} + \eta_j \, p^P_{kj} - \beta^S_j \left[p^S_{kj} - w_{kj} - c^S_j \right] = 0 \; \Leftrightarrow \; p^S_{kj} = \frac{\alpha_j + \eta_j \, p^P_{kj} + \beta^S_j \left[w_{kj} + c^S_j \right]}{2 \, \beta^S_j}. \tag{36}$$

where $w_{kj} > 0$ is the commission that Sj faces.

(35) and (36) imply that:

$$p^P_{kj} = \frac{\theta_j \, \alpha_j + \beta^P_j \, c^P_{kj} + w_{kj} \, \eta_j}{2 \, \beta^P_j} + \frac{\eta_j}{2 \, \beta^P_j} \, p^S_{kj}$$

$$= \frac{\theta_j \, \alpha_j + \beta^P_j \, c^P_{kj} + w_{kj} \, \eta_j}{2 \, \beta^P_j} + \frac{\eta_j}{2 \, \beta^P_j} \, \frac{\alpha_j + \eta_j \, p^P_{kj} + \beta^S_j \left[w_{kj} + c^S_j \right]}{2 \, \beta^S_j}$$

$$\Leftrightarrow \; p^P_{kj} \left[1 - \frac{\left(\eta_j \right)^2}{4 \, \beta^P_j \, \beta^S_j} \right] = \frac{\theta_j \, \alpha_j + \beta^P_j \, c^P_{kj} + w_{kj} \, \eta_j}{2 \, \beta^P_j} + \frac{\eta_j}{2 \, \beta^P_j} \, \frac{\alpha_j + \beta^S_j \left[w_{kj} + c^S_j \right]}{2 \, \beta^S_j}$$

$$\Leftrightarrow \; p^P_{kj} \left[\frac{4 \, \beta^P_j \, \beta^S_j - \left(\eta_j \right)^2}{4 \, \beta^P_j \, \beta^S_j} \right] = \frac{2 \, \beta^S_j \, \theta_j \, \alpha_j + 2 \, \beta^S_j \, \beta^P_j \, c^P_{kj} + 2 \, \beta^S_j \, w_{kj} \, \eta_j}{4 \, \beta^P_j \, \beta^S_j} + \frac{\eta_j \, \alpha_j + \eta_j \, \beta^S_j \left[w_{kj} + c^S_j \right]}{4 \, \beta^P_j \, \beta^S_j}$$

$$\Leftrightarrow \; p^P_{kj} = \frac{2 \, \beta^S_j \, \theta_j \, \alpha_j + 2 \, \beta^S_j \, \beta^P_j \, c^P_{kj} + 2 \, \beta^S_j \, w_{kj} \, \eta_j + \eta_j \, \alpha_j + \eta_j \, \beta^S_j \left[w_{kj} + c^S_j \right]}{4 \, \beta^P_j \, \beta^S_j - \left[\eta_j \right]^2}$$

$$\Leftrightarrow \; p^P_{kj} = \frac{\left[2 \, \beta^S_j \, \theta_j + \eta_j \right] \alpha_j + 2 \, \beta^S_j \, \beta^P_j \, c^P_{kj} + \eta_j \, \beta^S_j \, c^S_j + 3 \, \beta^S_j \, \eta_j \, w_{kj}}{\Phi_{2kj}}. \tag{37}$$

(37) reflects (14).

(36) and (37) imply that:

$$
\begin{aligned}
p_{kj}^S &= \frac{\alpha_j + \beta_j^S\, c_j^S}{2\,\beta_j^S} + \frac{\beta_j^S\, w_{kj}}{2\,\beta_j^S} + \frac{\eta_j}{2\,\beta_j^S}\, p_{kj}^P \\
&= \frac{\alpha_j + \beta_j^S\, c_j^S}{2\,\beta_j^S} + \frac{\beta_j^S\, w_{kj}}{2\,\beta_j^S} + \frac{\eta_j}{2\,\beta_j^S}\, \frac{\left[\, 2\,\beta_j^S\,\theta_j + \eta_j \,\right] \alpha_j + 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P + \eta_j\,\beta_j^S\, c_j^S \, + 3\,\beta_j^S\,\eta_j\, w_{kj}}{\Phi_{2kj}} \\
&= \frac{\alpha_j + \beta_j^S\, c_j^S}{2\,\beta_j^S} + \frac{\eta_j}{2\,\beta_j^S}\, \frac{\left[\, 2\,\beta_j^S\,\theta_j + \eta_j \,\right] \alpha_j + 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P + \eta_j\,\beta_j^S\, c_j^S}{\Phi_{2kj}} + \frac{\eta_j}{2\,\beta_j^S}\, \frac{3\,\beta_j^S\,\eta_j\, w_{kj}}{\Phi_{2kj}} + \frac{\beta_j^S\, w_{kj}}{2\,\beta_j^S} \\
&= \frac{\alpha_j + \beta_j^S\, c_j^S}{2\,\beta_j^S} + \frac{\eta_j}{2\,\beta_j^S}\, \frac{\left[\, 2\,\beta_j^S\,\theta_j + \eta_j \,\right] \alpha_j + 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P + \eta_j\,\beta_j^S\, c_j^S}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \\
&= \frac{\Phi_{2kj}\,\alpha_j + \Phi_{2kj}\,\beta_j^S\, c_j^S + \eta_j\left[\, 2\,\beta_j^S\,\theta_j + \eta_j \,\right]\alpha_j + \eta_j\, 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P + \left[\,\eta_j\,\right]^2 \beta_j^S\, c_j^S}{2\,\beta_j^S\,\Phi_{2kj}} \\
&\quad + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \\
&= \frac{\left[\,\Phi_{2kj} \, + 2\,\eta_j\,\beta_j^S\,\theta_j + \left(\eta_j\right)^2\right]\alpha_j + \left[\,\Phi_{2kj} + \left(\eta_j\right)^2\right]\beta_j^S\, c_j^S + \eta_j\, 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P}{2\,\beta_j^S\,\Phi_{2kj}} \\
&\quad + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \\
&= \frac{\left[\, 4\,\beta_j^P\,\beta_j^S + 2\,\eta_j\,\beta_j^S\,\theta_j \,\right]\alpha_j + 4\,\beta_j^P\,\beta_j^S\,\beta_j^S\, c_j^S + \eta_j\, 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P}{2\,\beta_j^S\,\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \\
&= \frac{2\,\beta_j^S\left[\, 2\,\beta_j^P + \eta_j\,\theta_j \,\right]\alpha_j + 4\,\beta_j^P\,\left[\,\beta_j^S\,\right]^2 c_j^S + \eta_j\, 2\,\beta_j^S\,\beta_j^P\, c_{kj}^P}{2\,\beta_j^S\,\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \\
&= \frac{\left[\, 2\,\beta_j^P + \eta_j\,\theta_j \,\right]\alpha_j + 2\,\beta_j^P\,\beta_j^S\, c_j^S + \eta_j\,\beta_j^P\, c_{kj}^P}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right].
\end{aligned}
\tag{38}
$$

(2), (37), (14), and (38) imply that consumers' initial demand for product j is:

$$
q_{kj}^S \;=\; \alpha_j - \beta_j^S \left\{ \frac{\left[\, 2\,\beta_j^P + \eta_j\,\theta_j \,\right]\alpha_j + 2\,\beta_j^P\,\beta_j^S\, c_j^S + \eta_j\,\beta_j^P\, c_{kj}^P}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,\left(\eta_j\right)^2}{\Phi_{2kj}} + 1\right] \right\}
$$

$$+ \eta_j \frac{\left[2\,\beta_j^S\,\theta_j + \eta_j \right] \alpha_j + 2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\beta_j^S\,c_j^S \; + 3\,\beta_j^S\,\eta_j\,w_{kj}}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j \right]^2}$$

$$= \alpha_j - \beta_j^S \frac{\left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j + 2\,\beta_j^P\,\beta_j^S\,c_j^S + \eta_j\,\beta_j^P\,c_{kj}^P}{\Phi_{2kj}} - \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\,\left(\eta_j \right)^2}{\Phi_{2kj}} + 1 \right]$$

$$+ \eta_j \frac{\left[2\,\beta_j^S\,\theta_j + \eta_j \right] \alpha_j + 2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\beta_j^S\,c_j^S}{\Phi_{2kj}} + \frac{\eta_j\,3\,\beta_j^S\,\eta_j\,w_{kj}}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j \right]^2}$$

$$= \alpha_j - \frac{\beta_j^S \left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j + \beta_j^S\,2\,\beta_j^P\,\beta_j^S\,c_j^S + \beta_j^S\,\eta_j\,\beta_j^P\,c_{kj}^P}{\Phi_{2kj}} - \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\,\left(\eta_j \right)^2}{\Phi_{2kj}} + 1 \right]$$

$$+ \frac{\eta_j \left[2\,\beta_j^S\,\theta_j + \eta_j \right] \alpha_j + \eta_j\,2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\eta_j\,\beta_j^S\,c_j^S}{\Phi_{2kj}} + \frac{\eta_j\,3\,\beta_j^S\,\eta_j\,w_{kj}}{\Phi_{2kj}}$$

$$= \frac{1}{\Phi_{2kj}} \{ \Phi_{2j}\,\alpha_j - \beta_j^S \left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j - \beta_j^S\,2\,\beta_j^P\,\beta_j^S\,c_j^S - \beta_j^S\,\eta_j\,\beta_j^P\,c_{kj}^P$$

$$+ \eta_j \left[2\,\beta_j^S\,\theta_j + \eta_j \right] \alpha_j + \eta_j\,2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\eta_j\,\beta_j^S\,c_j^S \}$$

$$+ \frac{3\,\left(\eta_j \right)^2\,\beta_j^S\,w_{kj}}{\Phi_{2kj}} - \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\,\left(\eta_j \right)^2}{\Phi_{2kj}} + 1 \right]$$

$$= \frac{1}{\Phi_{2kj}} \{ \left[4\,\beta_j^P\,\beta_j^S - \left(\eta_j \right)^2 \right] \alpha_j - \beta_j^S \left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j + \eta_j \left[2\,\beta_j^S\,\theta_j + \eta_j \right] \alpha_j$$

$$+ \eta_j\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \beta_j^S\,c_j^S \left[\left(\eta_j \right)^2 - 2\,\beta_j^S\,\beta_j^P \right] \}$$

$$+ \frac{3\,\left(\eta_j \right)^2\,\beta_j^S\,w_{kj}}{\Phi_{2kj}} - \frac{3\,\left(\eta_j \right)^2\,\beta_j^S\,w_{kj}}{2\,\Phi_{2kj}} - \frac{\beta_j^S\,w_{kj}}{2}$$

$$= \frac{1}{\Phi_{2kj}} \{ \left[4\,\beta_j^P\,\beta_j^S - \left(\eta_j \right)^2 - \beta_j^S\,2\,\beta_j^P - \beta_j^S\,\eta_j\,\theta_j \; + \eta_j\,2\,\beta_j^S\,\theta_j + \left(\eta_j \right)^2 \right] \alpha_j$$

$$+ \eta_j\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \beta_j^S\,c_j^S \left[\left(\eta_j \right)^2 - 2\,\beta_j^S\,\beta_j^P \right] \}$$

$$+ \frac{3\,\left(\eta_j \right)^2\,\beta_j^S\,w_{kj}}{2\,\Phi_{2kj}} - \frac{\beta_j^S\,w_{kj}}{2}$$

$$= \frac{1}{\Phi_{2kj}} \{ \left[2\,\beta_j^P\,\beta_j^S + \beta_j^S\,\eta_j\,\theta_j \right] \alpha_j + \eta_j\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \beta_j^S\,c_j^S \left[\left(\eta_j \right)^2 - 2\,\beta_j^S\,\beta_j^P \right] \}$$

$$+ \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\,\left(\eta_j \right)^2}{\Phi_{2kj}} - 1 \right]$$

$$= \frac{\beta_j^S}{\Phi_{2kj}} \left\{ \left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j + \eta_j\,\beta_j^P\,c_{kj}^P + c_j^S \left[\left(\eta_j \right)^2 - 2\,\beta_j^S\,\beta_j^P \right] \right\}$$

$$+ \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\,\left(\eta_j \right)^2}{\Phi_{2kj}} - 1 \right]. \tag{39}$$

Observe that:

$$\left[2\,\beta_j^P + \eta_j\,\theta_j \right] \alpha_j + \eta_j\,\beta_j^P\,c_{kj}^P + c_j^S \left[\left(\eta_j \right)^2 - 2\,\beta_j^S\,\beta_j^P \right]$$

$$= 2\,\beta_j^P\,\alpha_j + \eta_j\,\theta_j\,\alpha_j + \eta_j\,\beta_j^P\,c_{kj}^P + c_j^S\left[\eta_j\right]^2 - 2\,\beta_j^S\,\beta_j^P\,c_j^S$$

$$= 2\,\beta_j^P\,\alpha_j + \eta_j\,\theta_j\,\alpha_j - \eta_j\,\beta_j^P\,c_{kj}^P + 2\,\eta_j\,\beta_j^P\,c_{kj}^P + c_j^S\left[\eta_j\right]^2 - 2\,\beta_j^S\,\beta_j^P\,c_j^S$$

$$= \eta_j\,\theta_j\,\alpha_j - \eta_j\,\beta_j^P\,c_{kj}^P + c_j^S\left[\eta_j\right]^2 + 2\,\beta_j^P\,\alpha_j + 2\,\eta_j\,\beta_j^P\,c_{kj}^P - 2\,\beta_j^S\,\beta_j^P\,c_j^S$$

$$= \eta_j \left[\theta_j\,\alpha_j - \beta_j^P\,c_{kj}^P + c_j^S\,\eta_j \right] + 2\,\beta_j^P \left[\alpha_j + \eta_j\,c_{kj}^P - \beta_j^S\,c_j^S \right] = \eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^P\,\Delta_j. \tag{40}$$

(14), (39) and (40) imply that:

$$q_{kj}^S = \frac{\beta_j^S \left[\eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^P\,\Delta_{kj} \right]}{\Phi_{2kj}} + \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\left(\eta_j\right)^2}{\Phi_{2kj}} - 1 \right]$$

$$= \frac{\beta_j^S \left[\eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^P\,\Delta_{kj} \right]}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2} + \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\left(\eta_j\right)^2}{4\,\beta_j^P\,\beta_j^S - \left(\eta_j\right)^2} - 1 \right]$$

$$= \frac{\beta_j^S \left[\eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^P\,\Delta_{kj} \right]}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2} + \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{3\left(\eta_j\right)^2 - 4\,\beta_j^P\,\beta_j^S + \left(\eta_j\right)^2}{4\,\beta_j^P\,\beta_j^S - \left(\eta_j\right)^2} \right]$$

$$= \frac{\beta_j^S \left[\eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^P\,\Delta_{kj} \right]}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2} + \frac{\beta_j^S\,w_{kj}}{2} \left[\frac{4\left(\eta_j\right)^2 - 4\,\beta_j^P\,\beta_j^S}{4\,\beta_j^P\,\beta_j^S - \left(\eta_j\right)^2} \right]$$

$$= \frac{\beta_j^S\,\eta_j\,\overline{\Delta}_{kj} + 2\,\beta_j^S\,\beta_j^P\,\Delta_{kj}}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2} + 2\,\beta_j^S\,w_{kj} \left[\frac{\left(\eta_j\right)^2 - \beta_j^P\,\beta_j^S}{4\,\beta_j^P\,\beta_j^S - \left(\eta_j\right)^2} \right]$$

$$= \frac{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj}}{4 - \frac{\left[\eta_j\right]^2}{\beta_j^P\,\beta_j^S}} + 2\,\beta_j^S\,w_{kj} \left[\frac{\frac{\left(\eta_j\right)^2}{\beta_j^P\,\beta_j^S} - 1}{4 - \frac{\left(\eta_j\right)^2}{\beta_j^P\,\beta_j^S}} \right] = \frac{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj}}{4 - \Omega_j} + 2\,\beta_j^S\,w_{kj} \left[\frac{\Omega_j - 1}{4 - \Omega_j} \right]$$

$$= \frac{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - 2\,\beta_j^S \left[1 - \Omega_j \right] w_{kj}}{4 - \Omega_j}. \tag{41}$$

(15) implies (41) can be written as:

$$q^S_{kj} \;=\; \frac{\frac{\beta^S_j\,\Omega_j}{\eta_j}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - 2\,\beta^S_j\,[\,1-\Omega_j\,]\,w_{kj}}{4-\Omega_j}. \tag{42}$$

(11) and (42) imply that Sj's sales are:

$$Q^S_{kj} \;=\; \Theta_k\,q^S_{kj} \;=\; \frac{\Theta_k\left[\,\frac{\eta_j}{\beta^P_j}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - 2\,\beta^S_j\,(\,1-\Omega_j\,)\,w_{kj}\,\right]}{4-\Omega_j}.$$

(14) and (38) imply that:

$$\begin{aligned}
p^S_{kj}-w_{kj}-c^S_j \;&=\; \frac{\left[\,2\,\beta^P_j+\eta_j\,\theta_j\,\right]\alpha_j + 2\,\beta^P_j\,\beta^S_j\,c^S_j + \eta_j\,\beta^P_j\,c^P_{kj}}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,(\,\eta_j\,)^2}{\Phi_{2kj}} + 1\right] - w_{kj} - c^S_j \\
&= \frac{\left[\,2\,\beta^P_j+\eta_j\,\theta_j\,\right]\alpha_j + 2\,\beta^P_j\,\beta^S_j\,c^S_j + \eta_j\,\beta^P_j\,c^P_{kj} - \Phi_{2kj}\,c^S_j}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,(\,\eta_j\,)^2}{\Phi_{2kj}} + 1 - 2\right] \\
&= \frac{\left[\,2\,\beta^P_j+\eta_j\,\theta_j\,\right]\alpha_j + 2\,\beta^P_j\,\beta^S_j\,c^S_j + \eta_j\,\beta^P_j\,c^P_{kj} - \left[\,4\,\beta^P_j\,\beta^S_j - (\eta_j)^2\,\right]c^S_j}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,(\,\eta_j\,)^2}{\Phi_{2kj}} - 1\right] \\
&= \frac{\left[\,2\,\beta^P_j+\eta_j\,\theta_j\,\right]\alpha_j + \eta_j\,\beta^P_j\,c^P_{kj} + \left[\,(\eta_j)^2 - 2\,\beta^P_j\,\beta^S_j\,\right]c^S_j}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,(\,\eta_j\,)^2}{\Phi_{2kj}} - 1\right].
\end{aligned} \tag{43}$$

(14), (40) and (43) imply that:

$$\begin{aligned}
p^S_{kj}-w_{kj}-c^S_j \;&=\; \frac{\eta_j\,\overline{\Delta}_{kj} + 2\,\beta^P_j\,\Delta_{kj}}{\Phi_{2kj}} + \frac{w_{kj}}{2}\left[\frac{3\,(\eta_j)^2}{\Phi_{2kj}} - 1\right] \\
&= \frac{\eta_j\,\overline{\Delta}_{kj} + 2\,\beta^P_j\,\Delta_{kj}}{4\,\beta^P_j\,\beta^S_j - [\eta_j]^2} + \frac{w_{kj}}{2}\left[\frac{3\,(\eta_j)^2}{4\,\beta^P_j\,\beta^S_j - (\eta_j)^2} - 1\right] \\
&= \frac{\eta_j\,\overline{\Delta}_{kj} + 2\,\beta^P_j\,\Delta_{kj}}{4\,\beta^P_j\,\beta^S_j - [\eta_j]^2} + \frac{w_{kj}}{2}\left[\frac{3\,(\eta_j)^2 - 4\,\beta^P_j\,\beta^S_j + (\eta_j)^2}{4\,\beta^P_j\,\beta^S_j - (\eta_j)^2}\right] \\
&= \frac{\eta_j\,\overline{\Delta}_{kj} + 2\,\beta^P_j\,\Delta_{kj}}{4\,\beta^P_j\,\beta^S_j - [\eta_j]^2} + 2\,w_{kj}\left[\frac{(\eta_j)^2 - \beta^P_j\,\beta^S_j}{4\,\beta^P_j\,\beta^S_j - (\eta_j)^2}\right] \;=\; \frac{\frac{\eta_j}{\beta^P_j\,\beta^S_j}\,\overline{\Delta}_{kj} + \frac{2}{\beta^S_j}\,\Delta_{kj}}{4 - \frac{[\eta_j]^2}{\beta^P_j\,\beta^S_j}} + 2\,w_{kj}\left[\frac{\frac{(\eta_j)^2}{\beta^P_j\,\beta^S_j} - 1}{4 - \frac{(\eta_j)^2}{\beta^P_j\,\beta^S_j}}\right] \\
&= \frac{\frac{\Omega_j}{\eta_j}\,\overline{\Delta}_{kj} + \frac{2}{\beta^S_j}\,\Delta_{kj}}{4-\Omega_j} + 2\,w_{kj}\left[\frac{\Omega_j - 1}{4-\Omega_j}\right] \;=\; \frac{\frac{\Omega_j}{\eta_j}\,\overline{\Delta}_{kj} + \frac{2}{\beta^S_j}\,\Delta_{kj} - 2\,[\,1-\Omega_j\,]\,w_{kj}}{4-\Omega_j}.
\end{aligned} \tag{44}$$

(44) reflects (15).

(42) and (44) imply that:

$$p^S_{kj} - w_{kj} - c^S_j \;=\; \frac{q^S_{kj}}{\beta^S_j}. \tag{45}$$

(33), (42), and (45) imply that:

$$\pi_j \;=\; \left[\, p^S_{kj} - w_{kj} - c^S_j \,\right] \Theta_k\, q^S_{kj} \;=\; \frac{q^S_{kj}}{\beta^S_j} \Theta_k\, q^S_{kj} \;=\; \frac{\Theta_k}{\beta^S_j} \left[\, q^S_{kj} \,\right]^2$$

$$= \frac{\Theta_k}{\beta^S_j} \left[\frac{\frac{\beta^S_j\, \Omega_j}{\eta_j}\, \overline{\Delta}_{kj} + 2\, \Delta_{kj} - 2\, \beta^S_j \left[\, 1 - \Omega_j \,\right] w_{kj}}{4 - \Omega_j} \right]^2. \tag{46}$$

(1), (37), and (38) imply that

$$q^P_{kj} \;=\; \theta_j\, \alpha_j - \beta^P_j\, p^P_{kj} + \eta_j\, p^S_{kj}$$

$$= \theta_j\, \alpha_j - \beta^P_j\, \frac{\left[\, 2\, \beta^S_j\, \theta_j + \eta_j \,\right] \alpha_j + 2\, \beta^S_j\, \beta^P_j\, c^P_{kj} + \eta_j\, \beta^S_j\, c^S_j \;+ 3\, \beta^S_j\, \eta_j\, w_{kj}}{\Phi_{2j}}$$

$$+\; \eta_j\, \frac{\left[\, 2\, \beta^P_j + \eta_j\, \theta_j \,\right] \alpha_j + 2\, \beta^P_j\, \beta^S_j\, c^S_j + \eta_j\, \beta^P_j\, c^P_{kj}}{\Phi_{2j}} + \eta_j\, \frac{w_{kj}}{2} \left[\frac{3\, \left(\, \eta_j \,\right)^2}{\Phi_{2j}} + 1 \right]$$

$$= \frac{\Phi_{2j}\, \theta_j\, \alpha_j}{\Phi_{2j}} - \frac{\beta^P_j \left[\, 2\, \beta^S_j\, \theta_j + \eta_j \,\right] \alpha_j + \beta^P_j\, 2\, \beta^S_j\, \beta^P_j\, c^P_{kj} + \beta^P_j\, \eta_j\, \beta^S_j\, c^S_j}{\Phi_{2j}} - \frac{3\, \beta^P_j\, \beta^S_j\, \eta_j\, w_{kj}}{\Phi_{2j}}$$

$$+\; \frac{\eta_j \left[\, 2\, \beta^P_j + \eta_j\, \theta_j \,\right] \alpha_j + \eta_j\, 2\, \beta^P_j\, \beta^S_j\, c^S_j + \eta_j\, \eta_j\, \beta^P_j\, c^P_{kj}}{\Phi_{2j}} + \eta_j\, \frac{w_{kj}}{2} \left[\frac{3\, \left(\, \eta_j \,\right)^2}{\Phi_{2j}} + 1 \right]$$

$$= \frac{1}{\Phi_{2j}} \{\, \Phi_{2j}\, \theta_j\, \alpha_j - \beta^P_j \left[\, 2\, \beta^S_j\, \theta_j + \eta_j \,\right] \alpha_j - \beta^P_j\, 2\, \beta^S_j\, \beta^P_j\, c^P_{kj} - \beta^P_j\, \eta_j\, \beta^S_j\, c^S_j$$

$$+\; \eta_j \left[\, 2\, \beta^P_j + \eta_j\, \theta_j \,\right] \alpha_j + \eta_j\, 2\, \beta^P_j\, \beta^S_j\, c^S_j + \eta_j\, \eta_j\, \beta^P_j\, c^P_{kj} \,\}$$

$$+\; w_{kj} \left\{ \frac{\eta_j}{2} \left[\frac{3\, \left(\, \eta_j \,\right)^2}{\Phi_{2j}} + 1 \right] - \frac{3\, \beta^P_j\, \beta^S_j\, \eta_j}{\Phi_{2j}} \right\}$$

$$= \frac{1}{\Phi_{2j}} \{ \left[\, \Phi_{2j}\, \theta_j \;-\; \beta^P_j\, 2\, \beta^S_j\, \theta_j - \beta^P_j\, \eta_j \;+ \eta_j\, 2\, \beta^P_j + \left(\, \eta_j \,\right)^2\, \theta_j \,\right] \alpha_j$$

$$+\; \beta^P_j\, c^P_{kj} \left[\, \left(\eta_j\right)^2 \;-\; \beta^P_j\, 2\, \beta^S_j \,\right] + \eta_j\, \beta^P_j\, \beta^S_j\, c^S_j \,\}$$

$$+ \, w_{kj} \left\{ \frac{\eta_j}{2} \left[\frac{3 \, \left(\eta_j \right)^2 + \Phi_{2j}}{\Phi_{2j}} \right] - \frac{3 \, \beta_j^P \, \beta_j^S \, \eta_j}{\Phi_{2j}} \right\}$$

$$= \frac{1}{\Phi_{2j}} \{ \left[4 \, \beta_j^P \, \beta_j^S \, \theta_j - \left(\eta_j\right)^2 \, \theta_j - \beta_j^P \, 2 \, \beta_j^S \, \theta_j + \beta_j^P \, \eta_j \, + \left(\eta_j\right)^2 \, \theta_j \right] \alpha_j$$

$$+ \, \beta_j^P \, c_{kj}^P \left[\left(\eta_j\right)^2 - \beta_j^P \, 2 \, \beta_j^S \right] + \eta_j \, \beta_j^P \, \beta_j^S \, c_j^S \}$$

$$+ \, w_{kj} \left\{ \frac{\eta_j}{2} \left[\frac{3 \, \left(\eta_j \right)^2 + 4 \, \beta_j^P \, \beta_j^S - \left(\eta_j \right)^2}{\Phi_{2j}} \right] - \frac{3 \, \beta_j^P \, \beta_j^S \, \eta_j}{\Phi_{2j}} \right\}$$

$$= \frac{1}{\Phi_{2j}} \{ \left[2 \, \beta_j^P \, \beta_j^S \, \theta_j + \beta_j^P \, \eta_j \, \right] \alpha_j + \beta_j^P \, c_{kj}^P \left[\left(\eta_j \right)^2 - 2 \, \beta_j^P \, \beta_j^S \right] + \eta_j \, \beta_j^P \, \beta_j^S \, c_j^S \}$$

$$+ \, \eta_j \, w_{kj} \left\{ \frac{1}{2} \left[\frac{2 \, \left(\eta_j \right)^2 + 4 \, \beta_j^P \, \beta_j^S}{\Phi_{2j}} \right] - \frac{3 \, \beta_j^P \, \beta_j^S}{\Phi_{2j}} \right\}$$

$$= \frac{1}{\Phi_{2j}} \{ \beta_j^P \left[2 \, \beta_j^S \, \theta_j + \eta_j \right] \alpha_j + \beta_j^P \, c_{kj}^P \left[\left(\eta_j \right)^2 - 2 \, \beta_j^P \, \beta_j^S \right] + \eta_j \, \beta_j^P \, \beta_j^S \, c_j^S \}$$

$$+ \, \eta_j \, w_{kj} \left[\frac{\left(\eta_j \right)^2 + 2 \, \beta_j^P \, \beta_j^S}{\Phi_{2j}} - \frac{3 \, \beta_j^P \, \beta_j^S}{\Phi_{2j}} \right]$$

$$= \frac{\beta_j^P}{\Phi_{2j}} \{ \alpha_j \left[2 \, \beta_j^S \, \theta_j + \eta_j \right] + c_{kj}^P \left[\left(\eta_j \right)^2 - 2 \, \beta_j^P \, \beta_j^S \right] + \eta_j \, \beta_j^S \, c_j^S \} + \eta_j \, w_{kj} \left[\frac{\left(\eta_j \right)^2 - \beta_j^P \, \beta_j^S}{\Phi_{2j}} \right]. \tag{47}$$

Observe that:

$$\alpha_j \left[2 \, \beta_j^S \, \theta_j + \eta_j \right] + c_{kj}^P \left[\left(\eta_j \right)^2 - 2 \, \beta_j^P \, \beta_j^S \right] + \eta_j \, \beta_j^S \, c_j^S$$

$$= 2 \, \beta_j^S \, \theta_j \, \alpha_j + \eta_j \, \alpha_j + c_{kj}^P \left(\eta_j \right)^2 - 2 \, \beta_j^P \, \beta_j^S \, c_{kj}^P + 2 \, \eta_j \, \beta_j^S \, c_j^S - \eta_j \, \beta_j^S \, c_j^S$$

$$= 2 \, \beta_j^S \left[\theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P + \eta_j \, c_j^S \right] + \eta_j \left[\alpha_j + c_{kj}^P \, \eta_j - \beta_j^S \, c_j^S \right] \; = \; 2 \, \beta_j^S \, \overline{\Delta}_{kj} + \eta_j \, \Delta_{kj}. \tag{48}$$

(14), (15), (47) and (48) imply that:

$$q_{kj}^P \; = \; \frac{\beta_j^P \left[2 \, \beta_j^S \, \overline{\Delta}_{kj} + \eta_j \, \Delta_{kj} \right]}{\Phi_{2j}} + \eta_j \, w_{kj} \left[\frac{\left(\eta_j \right)^2 - \beta_j^P \, \beta_j^S}{\Phi_{2j}} \right]$$

$$= \frac{\beta_j^P \left[2 \, \beta_j^S \, \overline{\Delta}_{kj} + \eta_j \, \Delta_{kj} \right]}{4 \, \beta_j^P \, \beta_j^S - \left[\eta_j \right]^2} + \eta_j \, w_{kj} \left[\frac{\left(\eta_j \right)^2 - \beta_j^P \, \beta_j^S}{4 \, \beta_j^P \, \beta_j^S - \left(\eta_j \right)^2} \right]$$

$$= \frac{\frac{1}{\beta_j^S}\left[\,2\,\beta_j^S\,\overline{\Delta}_{kj} + \eta_j\,\Delta_{kj}\,\right]}{4 - \frac{[\eta_j]^2}{\beta_j^P\,\beta_j^S}} + \eta_j\,w_{kj}\left[\frac{\frac{(\eta_j)^2}{\beta_j^P\,\beta_j^S} - 1}{4 - \frac{(\eta_j)^2}{\beta_j^P\,\beta_j^S}}\right]$$

$$= \frac{2\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^S}\,\Delta_{kj}}{4 - \Omega_j} + \eta_j\,w_{kj}\left[\frac{\Omega_j - 1}{4 - \Omega_j}\right] = \frac{2\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\; - \eta_j\,w_{kj}\,[\,1 - \Omega_j\,]}{4 - \Omega_j}. \tag{49}$$

(11) and (49) imply that Pk's sales are:

$$Q_{kj}^P \;=\; \Theta_k\,q_{kj}^P \;=\; \frac{\Theta_k\left[\,2\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\; - \eta_j\,(\,1 - \Omega_j\,)\;w_{kj}\,\right]}{4 - \Omega_j}. \;\blacksquare \tag{50}$$

Lemma 5. *Suppose Sj competes against Pk when Sj sells on Pk ($j, k \in \{1, 2\}$). Then Pk's profit-maximizing commission for Sj is*

$$\frac{1}{2[\,1-\Omega_j\,]}\left[\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right].$$

<u>Proof</u>**.** Suppose Sj competes against Pk when Sj sells on Pk ($j, k \in \{1, 2\}$). Pk's profit is

$$\Pi_k \;=\; \left[\,p_{kj}^P - c_{kj}^P\,\right]\Theta_k\,q_{kj}^P - F + w_{kj}\,\Theta_k\,q_{kj}^S + \overline{\Pi}_{kl}, \tag{51}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from Sl ($j, k, l \in \{1, 2\}$, $j \neq l$), $\overline{\Pi}_{kl} > 0$ if Sl sells on Pk, and $\overline{\Pi}_{kl} = 0$ if Sl does not on Pk.[2]

(37) implies that

$$p_{kj}^P - c_{kj}^P \;=\; \frac{\left[\,2\,\beta_j^S\,\theta_j + \eta_j\,\right]\alpha_j + 2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\beta_j^S\,c_j^S\; + 3\,\beta_j^S\,\eta_j\,w_{kj}}{\Phi_{2j}} - c_{kj}^P$$

$$= \frac{\left[\,2\,\beta_j^S\,\theta_j + \eta_j\,\right]\alpha_j + 2\,\beta_j^S\,\beta_j^P\,c_{kj}^P + \eta_j\,\beta_j^S\,c_j^S\; + 3\,\beta_j^S\,\eta_j\,w_{kj} - 4\,\beta_j^P\,\beta_j^S\,c_{kj}^P + \left[\,\eta_j\,\right]^2\,c_{kj}^P}{\Phi_{2j}}$$

$$= \frac{\left[\,2\,\beta_j^S\,\theta_j + \eta_j\,\right]\alpha_j + c_{kj}^P\left[\,(\,\eta_j\,)^2 - 2\,\beta_j^S\,\beta_j^P\,\right] + \eta_j\,\beta_j^S\,c_j^S\; + 3\,\beta_j^S\,\eta_j\,w_{kj}}{\Phi_{2j}}. \tag{52}$$

(14), (15), (48) and (52) imply that:

$$p_{kj}^P - c_{kj}^P \;=\; \frac{2\,\beta_j^S\,\overline{\Delta}_{kj} + \eta_j\,\Delta_{kj} + 3\,\beta_j^S\,\eta_j\,w_{kj}}{\Phi_{2j}} \;=\; \frac{2\,\beta_j^S\,\overline{\Delta}_{kj} + \eta_j\,\Delta_{kj} + 3\,\beta_j^S\,\eta_j\,w_{kj}}{4\,\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2}$$

[2] $\overline{\Pi}_{kl}$ does not include w_{kj}.

$$= \frac{\frac{2}{\beta_j^P}\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^P\,\beta_j^S}\,\Delta_{kj} + \frac{3\,\eta_j}{\beta_j^P}\,w_{kj}}{4 - \frac{[\eta_j]^2}{\beta_j^P\,\beta_j^S}} \;=\; \frac{\frac{2}{\beta_j^P}\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^P\,\beta_j^S}\,\Delta_{kj} + \frac{3\,\eta_j}{\beta_j^P}\,w_{kj}}{4 - \Omega_j}. \tag{53}$$

(49) and (53) imply that:

$$\beta_{kj}^P\left[\,p_{kj}^P - c_{kj}^P\,\right] \;=\; \frac{2\,\overline{\Delta}_{kj} + \frac{\eta_j}{\beta_j^S}\,\Delta_{kj}}{4 - \Omega_j} + \frac{3\,\eta_j\,w_{kj}}{4 - \Omega_j} \;=\; q_{kj}^P + \frac{\eta_j\,w_{kj}\,[\,1 - \Omega_j\,]}{4 - \Omega_j} + \frac{3\,\eta_j\,w_{kj}}{4 - \Omega_j}$$

$$= \; q_{kj}^P + \frac{\eta_j\,w_{kj}\,[\,1 - \Omega_j + 3\,]}{4 - \Omega_j} \;=\; q_{kj}^P + \eta_j\,w_{kj}. \tag{54}$$

(54) implies that:

$$p_{kj}^P - c_{kj}^P \;=\; \frac{q_{kj}^P + \eta_j\,w_{kj}}{\beta_j^P}. \tag{55}$$

(51), (55) imply that:

$$\Pi_k \;=\; \left[\frac{q_{kj}^P + \eta_j\,w_{kj}}{\beta_j^P}\right]\Theta_k\,q_{kj}^P - F + w_{kj}\,\Theta_k\,q_{kj}^S + \overline{\Pi}_{kl}$$

$$= \; \frac{\Theta_k}{\beta_j^P}\left[\,q_{kj}^P\,\right]^2 + \frac{w_{kj}\,\Theta_k}{\beta_j^P}\,\eta_j\,q_{kj}^P - F + w_{kj}\,\Theta_k\,q_{kj}^S + \overline{\Pi}_{kl}$$

$$= \; \frac{\Theta_k}{\beta_j^P}\left[\,q_{kj}^P\,\right]^2 + w_{kj}\,\Theta_k\left[\frac{\eta_j}{\beta_j^P}\,q_{kj}^P + q_{kj}^S\right] - F + \overline{\Pi}_{kl}. \tag{56}$$

(56) implies that Pk chooses w_{kj} to

$$\text{Maximize }\; \Pi_k \;=\; \frac{\Theta_k}{\beta_j^P}\left[\,q_{kj}^P\,\right]^2 + w_{kj}\,\Theta_k\left[\frac{\eta_j}{\beta_j^P}\,q_{kj}^P + q_{kj}^S\right] - F + \overline{\Pi}_{kl} \;\;\Rightarrow\;\; \frac{\partial \Pi_k}{\partial w_{kj}} \;=\; 0, \tag{57}$$

where $\overline{\Pi}_{kl}$ is the profit that Pk secures from Sl ($j, k, l \in \{1, 2\}$, $j \neq l$), $\overline{\Pi}_{kl} > 0$ if Sl sells on Pk, $\overline{\Pi}_{kl} = 0$ if Sl does not on Pk, and $\overline{\Pi}_{kl}$ does not include w_{kj}.

(57) implies that:

$$\frac{2}{\beta_j^P}\,q_{kj}^P\,\frac{\partial q_{kj}^P}{\partial w_{kj}} + \frac{\eta_j}{\beta_j^P}\,q_{kj}^P + q_{kj}^S + w_{kj}\left[\frac{\eta_j}{\beta_j^P}\,\frac{\partial q_{kj}^P}{\partial w_{kj}} + \frac{\partial q_{kj}^S}{\partial w_{kj}}\right] \;=\; 0. \tag{58}$$

(41) and (49) imply that:

$$\frac{\partial q_{kj}^P}{\partial w_{kj}} \;=\; -\,\frac{\eta_j\,[\,1 - \Omega_j\,]}{4 - \Omega_j} \text{ and } \frac{\partial q_{kj}^S}{\partial w_{kj}} \;=\; -\,\frac{2\,\beta_j^S\,[\,1 - \Omega_j\,]}{4 - \Omega_j}. \tag{59}$$

(58) and (59) imply that:

$$-\frac{\eta_j\,[\,1-\Omega_j\,]}{4-\Omega_j}\,\frac{2}{\beta_j^P}\,q_{kj}^P+\frac{\eta_j}{\beta_j^P}\,q_{kj}^P+q_{kj}^S+w_{kj}\left[-\frac{\eta_j\,(\,1-\Omega_j\,)}{4-\Omega_j}\,\frac{\eta_j}{\beta_j^P}-\frac{2\,\beta_j^S\,(\,1-\Omega_j\,)}{4-\Omega_j}\right]\;=\;0$$

$$\Leftrightarrow\;\;\frac{\eta_j}{\beta_j^P}\left[1-\frac{2\,(\,1-\Omega_j\,)}{4-\Omega_j}\right]q_{kj}^P+q_{kj}^S-w_{kj}\,\frac{[\,1-\Omega_j\,]}{4-\Omega_j}\left[\frac{\left(\eta_j\right)^2}{\beta_j^P}+2\,\beta_j^S\right]\;=\;0$$

$$\Leftrightarrow\;\;\frac{\eta_j}{\beta_j^P}\left[\frac{4-\Omega_j-2+2\,\Omega_j}{4-\Omega_j}\right]q_{kj}^P+q_{kj}^S-w_{kj}\,\frac{[\,1-\Omega_j\,]}{4-\Omega_j}\left[\frac{\left(\eta_j\right)^2+2\,\beta_j^S\,\beta_j^P}{\beta_j^P}\right]\;=\;0$$

$$\Leftrightarrow\;\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]q_{kj}^P+q_{kj}^S-w_{kj}\,\frac{[\,1-\Omega_j\,]}{4-\Omega_j}\left[\frac{\frac{\left(\eta_j\right)^2}{\beta_j^S\,\beta_j^P}+2}{\frac{\beta_{kj}^P}{\beta_j^S\,\beta_j^P}}\right]\;=\;0$$

$$\Leftrightarrow\;\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]q_{kj}^P+q_{kj}^S-\beta_j^S\,w_{kj}\,\frac{[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]}{4-\Omega_j}\;=\;0$$

$$\Leftrightarrow\;\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]q_{kj}^P+q_{kj}^S\;=\;w_{kj}\,\frac{\beta_j^S\,[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]}{4-\Omega_j} \tag{60}$$

$$\Leftrightarrow\;\;w_{kj}\,\frac{\beta_j^S\,[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]}{4-\Omega_j}$$

$$=\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\frac{2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\;-\eta_j\,w_{kj}\,[\,1-\Omega_j\,]}{4-\Omega_j}+\frac{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}-2\,\beta_j^S\,[\,1-\Omega_j\,]\;w_{kj}}{4-\Omega_j} \tag{61}$$

$$\Leftrightarrow\;\;\beta_j^S\,[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]\,w_{kj}$$

$$=\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\left[2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\;-\eta_j\,w_{kj}\,(\,1-\Omega_j\,)\right]+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}-2\,\beta_j^S\,[\,1-\Omega_j\,]\;w_{kj}$$

$$\Leftrightarrow\;\;\beta_j^S\,[\,1-\Omega_j\,]\,[\,2+\Omega_j\,]\,w_{kj}+\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\eta_j\,w_{kj}\,[\,1-\Omega_j\,]+2\,\beta_j^S\,[\,1-\Omega_j\,]\;w_{kj}$$

$$=\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\left[2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\right]+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}$$

$$\Leftrightarrow\;\;w_{kj}\,[\,1-\Omega_j\,]\left[\beta_j^S\,(\,2+\Omega_j\,)+\frac{\left(\eta_j\right)^2}{\beta_j^P}\,\frac{2+\Omega_j}{4-\Omega_j}+2\,\beta_j^S\right]$$

$$=\;\frac{\eta_j}{\beta_j^P}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\,2\,\overline{\Delta}_{kj}+\frac{\left(\eta_j\right)^2}{\beta_j^P\,\beta_j^S}\left[\frac{2+\Omega_j}{4-\Omega_j}\right]\,\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}$$

$$\Leftrightarrow \quad w_{kj}\,[\,1-\Omega_j\,]\left[\beta_j^S\,(\,2+\Omega_j+2\,)+\frac{\big(\eta_j\big)^2}{\beta_j^P}\,\frac{2+\Omega_j}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[2\,\frac{2+\Omega_j}{4-\Omega_j}+1\right]+\Delta_{kj}\left[\frac{\big(\eta_j\big)^2}{\beta_j^P\,\beta_j^S}\,\frac{2+\Omega_j}{4-\Omega_j}+2\right]$$

$$\Leftrightarrow \quad w_{kj}\,[\,1-\Omega_j\,]\left[\beta_j^S\,(\,4+\Omega_j\,)+\frac{\big(\eta_j\big)^2}{\beta_j^P}\,\frac{2+\Omega_j}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{4+2\,\Omega_j\;+4-\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\Omega_j\,\frac{2+\Omega_j}{4-\Omega_j}+2\right] \tag{62}$$

$$\Leftrightarrow \quad w_{kj}\,[\,1-\Omega_j\,]\left[\beta_j^S\,(\,4+\Omega_j\,)+\frac{\big(\eta_j\big)^2}{\beta_j^P}\,\frac{2+\Omega_j}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{8+\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\frac{2\,\Omega_j+(\Omega_j)^2+8-2\,\Omega_j}{4-\Omega_j}\right]$$

$$\Leftrightarrow \quad w_{kj}\,[\,1-\Omega_j\,]\left[\beta_j^S\,(\,4+\Omega_j\,)+\beta_j^S\,\Omega_j\,\frac{2+\Omega_j}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{8+\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\frac{8+(\Omega_j)^2}{4-\Omega_j}\right] \tag{63}$$

$$\Leftrightarrow \quad \beta_j^S\,w_{kj}\,[\,1-\Omega_j\,]\left[4+\Omega_j+\frac{2\,\Omega_j+(\Omega_j)^2}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{8+\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\frac{8+(\Omega_j)^2}{4-\Omega_j}\right]$$

$$\Leftrightarrow \quad \beta_j^S\,w_{kj}\,[\,1-\Omega_j\,]\left[\frac{16-(\Omega_j)^2+2\,\Omega_j+(\Omega_j)^2}{4-\Omega_j}\right]$$

$$= \;\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{8+\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\frac{8+(\Omega_j)^2}{4-\Omega_j}\right]$$

$$\Leftrightarrow \quad 2\,\beta_j^S\,w_{kj}\,[\,1-\Omega_j\,]\left[\frac{8+\Omega_j}{4-\Omega_j}\right] \;=\; \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\left[\frac{8+\Omega_j}{4-\Omega_j}\right]+\Delta_{kj}\left[\frac{8+(\Omega_j)^2}{4-\Omega_j}\right]$$

$$\Leftrightarrow \quad 2\,\beta_j^S\,w_{kj}\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,] \;=\; \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\,[\,8+\Omega_j\,]+\Delta_j\,\big[\,8+(\Omega_j)^2\,\big]$$

$$\Leftrightarrow \quad w_{kj} \;=\; \frac{\eta_j}{2\,\beta_j^P\,\beta_j^S\,[\,1-\Omega_j\,]}\,\overline{\Delta}_{kj}+\frac{\big[\,8+(\Omega_j)^2\,\big]}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]}\,\Delta_{kj}$$

$$\Leftrightarrow \quad w_{kj} \;=\; \frac{\Omega_j}{2\,\eta_j\,[\,1-\Omega_j\,]}\,\overline{\Delta}_{kj} + \frac{\left[\,8+(\Omega_j)^2\,\right]}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]}\,\Delta_{kj} \tag{64}$$

$$\Leftrightarrow \quad w_{kj} \;=\; \frac{1}{2\,[\,1-\Omega_j\,]}\left\{\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}\right\}. \tag{65}$$

(60), (62), (63), and (64) reflect (15). (61) reflects (41) and (49). ■

Lemma 6. *Suppose Sj competes against Pk when Sj sells on Pk ($j,k \in \{1,2\}$). Then Sj's equilibrium output is $\frac{\Theta_k[\,2+\Omega_j\,]\,\Delta_{kj}}{8+\Omega_j}$, S$j$'s profit is $\frac{\Theta_k}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right]^2$, P$k$'s equilibrium output is $\frac{\Theta_k\,\overline{\Delta}_{kj}}{2}+\frac{\eta_j\Theta_k[\,2+\Omega_j\,]\,\Delta_{kj}}{2\beta_j^S[\,8+\Omega_j\,]}$, and P$k$'s profit from the commission it collects from Sj is $\Theta_k\,M_{kj} - F$. Pk sells more than Sj if Pk is a stronger seller than Sj (i.e., $Q_{kj}^P > Q_{kj}^S$ if $\frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > 1$).*

Proof. Suppose Sj competes against Pk when Sj sells on Pk ($j,k \in \{1,2\}$). Lemmas 4 and 5 imply that Sj's sales are:

$$\begin{aligned}
Q_{kj}^S \;&=\; \frac{\Theta_k\left[\,\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,2\,\beta_j^S\,[\,1-\Omega_j\,]}{4-\Omega_j}\,w_{kj} \\
&=\; \frac{\Theta_k\left[\,\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,2\,\beta_j^S\,[\,1-\Omega_j\,]}{2\,[\,1-\Omega_j\,]\,[\,4-\Omega_j\,]}\left\{\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right\} \\
&=\; \frac{\Theta_k\left[\,\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}+2\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,\beta_j^S}{4-\Omega_j}\left\{\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right\} \\
&=\; \frac{\Theta_k}{4-\Omega_j}\left\{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - \beta_j^S\left[\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right]\right\} \\
&=\; \frac{\Theta_k}{4-\Omega_j}\left\{\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} + 2\,\Delta_{kj} - \frac{\beta_j^S\,\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} - \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{8+\Omega_j}\right\} \\
&=\; \frac{\Theta_k}{4-\Omega_j}\left\{\overline{\Delta}_{kj}\left[\frac{\eta_j}{\beta_j^P} - \frac{\beta_j^S\,\Omega_j}{\eta_j}\right] + \Delta_{kj}\left[2-\frac{8+(\Omega_j)^2}{8+\Omega_j}\right]\right\} \\
&=\; \frac{\Theta_k}{4-\Omega_j}\,\Delta_{kj}\left[2-\frac{8+(\Omega_j)^2}{8+\Omega_j}\right] \;=\; \frac{\Theta_k}{4-\Omega_j}\,\Delta_{kj}\left[\frac{16+2\,\Omega_j-8-(\Omega_j)^2}{8+\Omega_j}\right]
\end{aligned} \tag{66}$$

$$= \frac{\Theta_k}{4-\Omega_j}\,\Delta_{kj}\left[\frac{8+2\,\Omega_j-(\Omega_j)^2}{8+\Omega_j}\right] \;=\; \frac{\Theta_k}{4-\Omega_j}\,\Delta_{kj}\,\frac{[\,4-\Omega_j\,]\,[\,2+\Omega_j\,]}{8+\Omega_j} \;=\; \frac{\Theta_k\,[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}. \tag{67}$$

The first equality in (66) reflects $\frac{\eta_j}{\beta_j^P} = \frac{\beta_j^S\,\Omega_j}{\eta_j}$ because (15) implies that $\frac{\beta_j^S\,\Omega_j}{\eta_j} = \frac{\beta_j^S[\eta_j]^2}{\eta_j\,\beta_j^P\,\beta_j^S} = \frac{\eta_j}{\beta_j^P}$.

(67) implies that

$$q_{kj}^S \;=\; \frac{[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}. \tag{68}$$

Lemma 4 and (68) imply that Sj's profit is

$$\pi_j^S \;=\; \frac{\Theta_k}{\beta_j^S}\left[\,q_{kj}^S\,\right]^2 \;=\; \frac{\Theta_k}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]^2. \tag{69}$$

Lemmas 4 and 5 imply that Pk's sales are:

$$Q_{kj}^P \;=\; \frac{\Theta_k\left[\,2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,\eta_j\,[\,1-\Omega_j\,]}{4-\Omega_j}\,w_{kj}$$

$$= \frac{\Theta_k\left[\,2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,\eta_j\,[\,1-\Omega_j\,]}{2\,[\,1-\Omega_j\,]\,[\,4-\Omega_j\,]}\left\{\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j}+\frac{\left[\,8+(\Omega_j)^2\,\right]\;\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right\}$$

$$= \frac{\Theta_k\left[\,2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,\eta_j}{2\,[\,4-\Omega_j\,]}\left\{\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j}+\frac{\left[\,8+(\Omega_j)^2\,\right]\;\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right\}$$

$$= \frac{\Theta_k\left[\,2\,\overline{\Delta}_{kj}+\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}\right]}{4-\Omega_j} - \frac{\Theta_k\,\eta_j}{2\,[\,4-\Omega_j\,]}\,\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} - \frac{\Theta_k\,\eta_j}{2\,[\,4-\Omega_j\,]}\,\frac{\left[\,8+(\Omega_j)^2\,\right]\;\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}$$

$$= \frac{2\,\Theta_k\,\overline{\Delta}_{kj}}{4-\Omega_j} + \frac{\Theta_k\,\frac{\eta_j}{\beta_j^S}\,\Delta_{kj}}{4-\Omega_j} - \frac{\Theta_k\,\Omega_j\,\overline{\Delta}_{kj}}{2\,[\,4-\Omega_j\,]} - \frac{\Theta_k\,\eta_j\left[\,8+(\Omega_j)^2\,\right]\;\Delta_{kj}}{2\,\beta_j^S\,[\,4-\Omega_j\,]\,[\,8+\Omega_j\,]}$$

$$= \frac{\Theta_k\,\overline{\Delta}_{kj}}{4-\Omega_j}\left[2-\frac{\Omega_j}{2}\right] + \frac{\eta_j\,\Theta_k\,\Delta_{kj}}{\beta_j^S\,[\,4-\Omega_j\,]}\left[1-\frac{8+(\Omega_j)^2}{2\,(\,8+\Omega_j\,)}\right]$$

$$= \frac{\Theta_k\,\overline{\Delta}_{kj}}{4-\Omega_j}\left[\frac{4-\Omega_j}{2}\right] + \frac{\eta_j\,\Theta_k\,\Delta_{kj}}{\beta_j^S\,[\,4-\Omega_j\,]}\left[\frac{16+2\,\Omega_j-8-(\Omega_j)^2}{2\,(\,8+\Omega_j\,)}\right]$$

$$= \frac{\Theta_k\,\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,\Theta_k\,\Delta_{kj}}{\beta_j^S\,[\,4-\Omega_j\,]}\left[\frac{8+2\,\Omega_j-(\Omega_j)^2}{2\,(\,8+\Omega_j\,)}\right]$$

$$= \frac{\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j \Theta_k \Delta_{kj}}{\beta_j^S [4 - \Omega_j]} \frac{[4 - \Omega_j][2 + \Omega_j]}{2[8 + \Omega_j]} = \frac{\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j \Theta_k [2 + \Omega_j] \Delta_{kj}}{2\beta_j^S [8 + \Omega_j]}. \quad (70)$$

(70) implies that:

$$q_{kj}^P = \frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j [2 + \Omega_j] \Delta_{kj}}{2 \beta_j^S [8 + \Omega_j]}. \quad (71)$$

(68) and (70) imply that:

$$Q_{kj}^P = \frac{\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j}{2 \beta_j^S} Q_{kj}^S. \quad (72)$$

(67) and (72) imply that

$$Q_{kj}^P > Q_{kj}^S \Leftrightarrow \frac{\Theta_k \overline{\Delta}_{kj}}{2} + \frac{\eta_j}{2 \beta_j^S} Q_{kj}^S > Q_{kj}^S \Leftrightarrow \frac{\Theta_k \overline{\Delta}_{kj}}{2} > Q_{kj}^S \left[1 - \frac{\eta_j}{2 \beta_j^S} \right]$$

$$\Leftrightarrow \frac{\Theta_k \overline{\Delta}_{kj}}{2} > \frac{\Theta_k [2 + \Omega_j] \Delta_{kj}}{8 + \Omega_j} \left[1 - \frac{\eta_j}{2 \beta_j^S} \right] \Leftrightarrow \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > \frac{2[2 + \Omega_j]}{8 + \Omega_j} \left[1 - \frac{\eta_j}{2 \beta_j^S} \right]$$

$$\Leftrightarrow \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > \frac{2 + \Omega_j}{8 + \Omega_j} \left[2 - \frac{\eta_j}{\beta_j^S} \right]. \quad (73)$$

Observe that:

$$\frac{\partial \left(\frac{2+\Omega_j}{8+\Omega_j} \right)}{\partial \Omega_j} = \frac{8 + \Omega_j - [2 + \Omega_j]}{[8 + \Omega_j]^2} = \frac{6}{[8 + \Omega_j]^2} > 0. \quad (74)$$

(74) implies that for $\Omega_j \in (0, 1)$,

$$\frac{2 + \Omega_j}{8 + \Omega_j} < \max \frac{2 + \Omega_j}{8 + \Omega_j} = \frac{2 + 1}{8 + 1} = \frac{1}{3}. \quad (75)$$

Because $2 - \frac{\eta_j}{\beta_j^S} < 2$, (75) implies that for $\Omega_j \in (0, 1)$,

$$\frac{2 + \Omega_j}{8 + \Omega_j} \left[2 - \frac{\eta_j}{\beta_j^S} \right] < \frac{2}{3}. \quad (76)$$

(73) and (76) imply that for $\Omega_j \in (0, 1)$,

$$Q_{kj}^P > Q_{kj}^S \text{ if } \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > 1. \quad (77)$$

(56), (68), (71), and Lemma 5 imply that Pk's profit from the commission it collects from Sj and from entering Sj's product market is:

$$\Pi_k = \frac{\Theta_k}{\beta_j^P} \left[q_{kj}^P \right]^2 + w_{kj}\,\Theta_k \left[\frac{\eta_j}{\beta_j^P}\, q_{kj}^P + q_{kj}^S \right] - F$$

$$= \frac{\Theta_k}{\beta_j^P} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2\,\beta_j^S \left[8 + \Omega_j \right]} \right]^2$$

$$+ \frac{\Theta_k}{2\left[1 - \Omega_j \right]} \left\{ \frac{\Omega_j\, \overline{\Delta}_{kj}}{\eta_j} + \frac{\left[8 + (\Omega_j)^2 \right] \Delta_{kj}}{\beta_j^S \left(8 + \Omega_j \right)} \right\}$$

$$\cdot \left\{ \frac{\eta_j}{\beta_j^P} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left(2 + \Omega_j \right) \Delta_{kj}}{2\,\beta_j^S \left(8 + \Omega_j \right)} \right] + \frac{\left[2 + \Omega_j \right] \Delta_{kj}}{8 + \Omega_j} \right\} - F$$

$$= \frac{\Theta_k}{\beta_j^P} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2\,\beta_j^S \left[8 + \Omega_j \right]} \right]^2$$

$$+ \frac{\Theta_k}{2\left[1 - \Omega_j \right]} \left\{ \frac{\Omega_j\, \overline{\Delta}_{kj}}{\eta_j} + \frac{\left[8 + (\Omega_j)^2 \right] \Delta_{kj}}{\beta_j^S \left(8 + \Omega_j \right)} \right\}$$

$$\cdot \left[\frac{\eta_j}{\beta_j^P} \frac{\overline{\Delta}_{kj}}{2} + \frac{\left(\eta_j \right)^2}{\beta_j^P\, \beta_j^S} \frac{\left(2 + \Omega_j \right) \Delta_{kj}}{2 \left(8 + \Omega_j \right)} + \frac{\left(2 + \Omega_j \right) \Delta_{kj}}{8 + \Omega_j} \right] - F$$

$$= \frac{\Theta_k}{\beta_j^P} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2\,\beta_j^S \left[8 + \Omega_j \right]} \right]^2$$

$$+ \frac{\Theta_k}{2\left[1 - \Omega_j \right]} \left\{ \frac{\Omega_j\, \overline{\Delta}_{kj}}{\eta_j} + \frac{\left[8 + (\Omega_j)^2 \right] \Delta_{kj}}{\beta_j^S \left(8 + \Omega_j \right)} \right\}$$

$$\cdot \left[\frac{\eta_j}{\beta_j^P} \frac{\overline{\Delta}_{kj}}{2} + \frac{\Omega_j \left(2 + \Omega_j \right) \Delta_{kj}}{2 \left(8 + \Omega_j \right)} + \frac{\left(2 + \Omega_j \right) \Delta_{kj}}{8 + \Omega_j} \right] - F \tag{78}$$

$$= \frac{\Theta_k}{\beta_j^P} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2\,\beta_j^S \left[8 + \Omega_j \right]} \right]^2$$

$$+ \frac{\Theta_k}{2\left[1 - \Omega_j \right]} \left\{ \frac{\Omega_j\, \overline{\Delta}_{kj}}{\eta_j} + \frac{\left[8 + (\Omega_j)^2 \right] \Delta_{kj}}{\beta_j^S \left(8 + \Omega_j \right)} \right\} \left[\frac{\eta_j}{\beta_j^P} \frac{\overline{\Delta}_{kj}}{2} + \frac{3\,\Omega_j}{2} \frac{\left(2 + \Omega_j \right) \Delta_{kj}}{\left(8 + \Omega_j \right)} \right] - F$$

$$= \frac{\Theta_k}{\beta_j^P} \left[\frac{\left(\overline{\Delta}_{kj} \right)^2}{4} + \frac{\left(\eta_j \right)^2 \left(2 + \Omega_j \right)^2 \left(\Delta_{kj} \right)^2}{4 \left(\beta_j^S \right)^2 \left(8 + \Omega_j \right)^2} + 2\, \frac{\overline{\Delta}_{kj}}{2} \frac{\eta_j \left[2 + \Omega_j \right] \Delta_{kj}}{2\,\beta_j^S \left[8 + \Omega_j \right]} \right] - F$$

$$
\begin{aligned}
& + \frac{\Theta_k}{2\,[\,1-\Omega_j\,]} \left\{ \frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j}\,\frac{\eta_j}{\beta_j^P}\,\frac{\overline{\Delta}_{kj}}{2} + \frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j}\,\frac{3\,\Omega_j}{2}\,\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{(\,8+\Omega_j\,)} \right. \\
& \qquad \left. + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\,\frac{\eta_j}{\beta_j^P}\,\frac{\overline{\Delta}_{kj}}{2} + \frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\,\frac{3\,\Omega_j}{2}\,\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{(\,8+\Omega_j\,)} \right\} \\
= \; & \frac{\Theta_k}{\beta_j^P} \left[\frac{\left(\overline{\Delta}_{kj}\right)^2}{4} + \frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{4\,\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)} \right] - F \\
& + \frac{\Theta_k}{2\,[\,1-\Omega_j\,]} \left\{ \frac{\Omega_j\;\left(\overline{\Delta}_{kj}\right)^2}{2\,\beta_j^P} + \frac{3\,(\Omega_j)^2\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\eta_j\,(\,8+\Omega_j\,)} \right. \\
& \qquad \left. + \frac{\eta_j\left[\,8+(\Omega_j)^2\,\right]\;\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\beta_j^S\,\beta_j^P\,(\,8+\Omega_j\,)} + \frac{3\,\Omega_j\,(\,2+\Omega_j\,)\left[\,8+(\Omega_j)^2\,\right]\;(\Delta_{kj})^2}{2\,\beta_j^S\,(\,8+\Omega_j\,)^2} \right\} \\
= \; & \Theta_k \left[\frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P} + \frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{4\,\beta_j^P\;\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\beta_j^P\,\beta_j^S\,(\,8+\Omega_j\,)} \right] - F \\
& + \Theta_k \left\{ \frac{\Omega_j\;\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(\,1-\Omega_j\,)} + \frac{3\,(\Omega_j)^2\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\eta_j\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)} \right. \\
& \qquad \left. + \frac{\eta_j\left[\,8+(\Omega_j)^2\,\right]\;\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\beta_j^S\,\beta_j^P\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)} + \frac{3\,\Omega_j\,(\,2+\Omega_j\,)\left[\,8+(\Omega_j)^2\,\right]\;(\Delta_{kj})^2}{4\,\beta_j^S\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)^2} \right\} \\
= \; & \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P} + \frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{4\,\beta_j^P\;\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\beta_j^P\,\beta_j^S\,(\,8+\Omega_j\,)} \right. \\
& \qquad + \frac{\Omega_j\;\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(\,1-\Omega_j\,)} + \frac{3\,(\Omega_j)^2\,(\,2+\Omega_j\,)\;\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\eta_j\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)} \\
& \qquad \left. + \frac{\eta_j\left[\,8+(\Omega_j)^2\,\right]\;\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\beta_j^S\,\beta_j^P\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)} + \frac{3\,\Omega_j\,(\,2+\Omega_j\,)\left[\,8+(\Omega_j)^2\,\right]\;(\Delta_{kj})^2}{4\,\beta_j^S\,(\,1-\Omega_j\,)\,(\,8+\Omega_j\,)^2} \right\} - F \\
= \; & \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P} \left[1 + \frac{\Omega_j}{1-\Omega_j} \right] + \frac{(\Delta_{kj})^2\,(\,2+\Omega_j\,)}{4\,\beta_j^S\,(\,8+\Omega_j\,)^2} \left[\frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)}{\beta_j^P\,\beta_j^S} + \frac{3\,\Omega_j\left[\,8+(\Omega_j)^2\,\right]}{1-\Omega_j} \right] \right. \\
& \qquad \left. + \frac{\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,(\,8+\Omega_j\,)} \left[\frac{\eta_j\,(\,2+\Omega_j\,)}{\beta_j^P\,\beta_j^S} + \frac{3\,(\Omega_j)^2\,(\,2+\Omega_j\,)}{2\,\eta_j\,(\,1-\Omega_j\,)} + \frac{\eta_j\left[\,8+(\Omega_j)^2\,\right]}{2\,\beta_j^S\,\beta_j^P\,(\,1-\Omega_j\,)} \right] \right\} - F \\
= \; & \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(\,1-\Omega_j\,)} + \frac{(\Delta_{kj})^2\,(\,2+\Omega_j\,)}{4\,\beta_j^S\,(\,8+\Omega_j\,)^2} \left[\Omega_j\,(\,2+\Omega_j\,) + \frac{3\,\Omega_j\left[\,8+(\Omega_j)^2\,\right]}{1-\Omega_j} \right] \right. \\
& \qquad \left. + \frac{\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,(\,8+\Omega_j\,)} \left[\frac{\Omega_j\,(\,2+\Omega_j\,)}{\eta_j} + \frac{3\,(\Omega_j)^2\,(\,2+\Omega_j\,)}{2\,\eta_j\,(\,1-\Omega_j\,)} + \frac{\Omega_j\left[\,8+(\Omega_j)^2\,\right]}{2\,\eta_j\,(\,1-\Omega_j\,)} \right] \right\} - F \qquad (79)
\end{aligned}
$$

$$= \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(1-\Omega_j)} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{4\,\beta_j^S\,(8+\Omega_j)^2} \left[2+\Omega_j + \frac{3\left[8+(\Omega_j)^2\right]}{1-\Omega_j} \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\eta_j\,(8+\Omega_j)} \left[2+\Omega_j + \frac{3\,\Omega_j\,(2+\Omega_j)}{2\,(1-\Omega_j)} + \frac{8+(\Omega_j)^2}{2\,(1-\Omega_j)} \right] \right\} - F$$

$$= \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(1-\Omega_j)} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{4\,\beta_j^S\,(8+\Omega_j)^2} \left[\frac{(2+\Omega_j)(1-\Omega_j)+24+3\,(\Omega_j)^2}{1-\Omega_j} \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{2\,\eta_j\,(8+\Omega_j)} \left[\frac{2\,(1-\Omega_j)(2+\Omega_j)+6\,\Omega_j+3\,(\Omega_j)^2+8+(\Omega_j)^2}{2\,(1-\Omega_j)} \right] \right\} - F$$

$$= \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(1-\Omega_j)} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{4\,\beta_j^S\,(8+\Omega_j)^2\,(1-\Omega_j)} \left[2-(\Omega_j)^2-\Omega_j+24+3\,(\Omega_j)^2 \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\eta_j\,(8+\Omega_j)(1-\Omega_j)} \left[2\left(2-(\Omega_j)^2-\Omega_j\right)+6\,\Omega_j+4\,(\Omega_j)^2+8 \right] \right\} - F$$

$$= \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(1-\Omega_j)} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{4\,\beta_j^S\,(8+\Omega_j)^2\,(1-\Omega_j)} \left[26-\Omega_j+2\,(\Omega_j)^2 \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\eta_j\,(8+\Omega_j)(1-\Omega_j)} \left[4-2\,(\Omega_j)^2-2\,\Omega_j+6\,\Omega_j+4\,(\Omega_j)^2+8 \right] \right\} - F$$

$$= \Theta_k \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{4\,\beta_j^P\,(1-\Omega_j)} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{4\,\beta_j^S\,(8+\Omega_j)^2\,(1-\Omega_j)} \left[26-\Omega_j+2\,(\Omega_j)^2 \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{4\,\eta_j\,(8+\Omega_j)(1-\Omega_j)} \left[12+4\,\Omega_j+2\,(\Omega_j)^2 \right] \right\} - F$$

$$= \frac{\Theta_k}{2\,(1-\Omega_j)} \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{2\,\beta_j^P} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{2\,\beta_j^S\,(8+\Omega_j)^2} \left[26-\Omega_j+2\,(\Omega_j)^2 \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{\eta_j\,(8+\Omega_j)} \left[6+2\,\Omega_j+(\Omega_j)^2 \right] \right\} - F$$

$$= \frac{\Theta_k}{2\,(1-\Omega_j)} \left\{ \frac{\left(\overline{\Delta}_{kj}\right)^2}{2\,\beta_j^P} + \frac{(\Delta_{kj})^2\,(2+\Omega_j)\,\Omega_j}{2\,\beta_j^S\,(8+\Omega_j)^2} \left[26-\Omega_j+2\,(\Omega_j)^2 \right] \right.$$

$$\left. + \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{\eta_j\,(8+\Omega_j)} \left[6+2\,\Omega_j+(\Omega_j)^2 \right] \right\} - F.$$

(78) and (79) reflect (15). ■

Proposition 1. *Suppose Condition FS holds. In the monopolistic platform setting, both* *sellers sell on P and P enters both sellers' product markets in equilibrium. Sj's equilibrium profit is* $\frac{\Theta}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{Pj}}{8+\Omega_j}\right]^2$*, and P's equilibrium profit is* $\Theta\, M_{P1} - F + \Theta\, M_{P2} - F$.

Proof. Suppose Condition FS holds. Lemma 3 implies that P's profit is $\frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{8\, b_j^S}$, if S$j$ ($j \in \{1,2\}$) sells product j on P and P does not enter Sj's product market. Lemma 6 implies that P's profit is $\Theta\, M_{Pj} - F$, if Sj ($j \in \{1,2\}$) sells product j on P and P enters Sj's product market. Because Condition FS holds, $\Theta\, M_{Pj} - F > \frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{8\, b_j^S}$, i.e., P secures a higher profit by entering Sj's market than "no entry". Therefore, if Sj sells on P, P will enter Sj's market, Lemma 6 implies that P's profit is $\Theta\, M_{Pj} - F$ and Sj's profit is $\frac{\Theta}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{Pj}}{8+\Omega_j}\right]^2$.

Therefore, knowing P's entry decisions, Sj ($j \in \{1,2\}$) will choose to sell on P because he secures a positive profit if he sells on P (i.e., $\frac{\Theta}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{Pj}}{8+\Omega_j}\right]^2 > 0$) while he secures zero profit if he does not sell on P, regardless of the other seller's choice.

Therefore, in equilibrium, both S1 and S2 sell on P, and P enters each seller's market. Lemma 6 implies Sj's profit is $\frac{\Theta}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{Pj}}{8+\Omega_j}\right]^2$, and P's profit is $\Theta\, M_{P1} - F + \Theta\, M_{P2} - F$. ■

Lemma 7. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$. *Further suppose both platforms commit not to enter. Then Sj (*$j \in \{1,2\}$*) is indifferent between selling on P1 and selling on P2 if* $\frac{\Theta_1}{\Theta_2} = 1$*, whereas Sj sells on P1 if* $\frac{\Theta_1}{\Theta_2} > 1$.

Proof. Lemma 3 implies that Sj's profit is $\frac{\Theta_k\left[\widetilde{\Delta}_{kj}\right]^2}{16\, b_j^S}$ if Sj sells on Pk ($j, k \in \{1,2\}$). (12) implies that

$$\widetilde{\Delta}_{kj} \;=\; \Delta_{kj} + \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj} \;=\; \alpha_j - \beta_j^S\, c_j^S + \eta_j\, c_{kj}^P + \frac{\eta_j}{\beta_j^P}\left[\,\theta_j\,\alpha_j - \beta_j^P\, c_{kj}^P + \eta_j\, c_j^S\,\right] \qquad (80)$$

(80) implies that

$$\frac{\partial \widetilde{\Delta}_{kj}}{\partial c_{kj}^P} \;=\; \eta_j - \frac{\eta_j}{\beta_j^P}\,\beta_j^P \;=\; 0. \qquad (81)$$

(81) implies that $\widetilde{\Delta}_{1j} = \widetilde{\Delta}_{2j}$. Consequently, if $\Theta_1 = \Theta_2$, then Sj is indifferent between selling on P1 and selling on P2; if $\Theta_1 > \Theta_2$, then Sj sells on P1. ■

Lemma 8. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$. *Further suppose platforms both make no commitment. If* $\frac{c_{2j}^P}{c_{1j}^P} < 1$*, then Sj will sell on P1. If* $\frac{c_{2j}^P}{c_{1j}^P} > 1$*, then Sj will: (i) sell on P1 when* $\frac{\Theta_1}{\Theta_2} > \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$*; and (ii) sell on P2 when* $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$.

Proof. Lemma 6 implies that Sj's profit is $\frac{\Theta_k}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j}\right]^2$ if Sj sells on Pk ($j,k \in \{1,2\}$). Therefore,

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{1j}}{8+\Omega_j}\right]^2 \gtreqless \frac{\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j}\right]^2 \Leftrightarrow \frac{\Theta_1}{\Theta_2} \gtreqless \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2.$$

Observe that

$$\frac{\Delta_{2j}}{\Delta_{1j}} \gtreqless 1 \Leftrightarrow \frac{c_{2j}^P}{c_{1j}^P} \gtreqless 1.$$

First suppose $\frac{c_{2j}^P}{c_{1j}^P} < 1$. Then S$j$ will sell on P1 because $\frac{\Theta_1}{\Theta_2} \geq 1 > \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$.

Next suppose $\frac{c_{2j}^P}{c_{1j}^P} > 1$. Then S$j$ will: (i) sell on P1 when $\frac{\Theta_1}{\Theta_2} > \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$; and (ii) sell on P2 when $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{2j}}{\Delta_{1j}}\right]^2$. ■

Lemma 9. *Suppose Condition FS holds and $\frac{\Theta_1}{\Theta_2} \geq 1$. Further suppose P1 commits not to enter and P2 makes no commitment. Then Sj ($j \in \{1,2\}$) will sell on P1.*

Proof. Condition FS ensures that P2 will enter Sj's market if Sj sells on P2 ($j \in \{1,2\}$). Lemma 3 implies that Sj's profit is $\frac{\Theta_1[\widetilde{\Delta}_{1j}]^2}{16\,b_j^S}$ if Sj sells on P1. Lemma 6 implies that Sj's profit is $\frac{\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j}\right]^2$ if Sj sells on P2. (10) implies that:

$$\frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\,b_j^S} > \frac{\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j}\right]^2 \Leftrightarrow \frac{\Theta_1\left[\widetilde{\Delta}_{1j}\right]^2}{16\,\beta_j^S\,[1-\Omega_j]} > \frac{\Theta_2}{\beta_j^S}\left[\frac{(2+\Omega_j)\,\Delta_{2j}}{8+\Omega_j}\right]^2$$

$$\Leftrightarrow \frac{\Theta_1}{\Theta_2} > \left[\frac{4\sqrt{1-\Omega_j}\,(2+\Omega_j)}{8+\Omega_j}\frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}}\right]^2. \tag{82}$$

(82) holds because

$$\frac{\Theta_1}{\Theta_2} \geq 1 > \left[\frac{4\sqrt{1-\Omega_j}\,(2+\Omega_j)}{8+\Omega_j}\frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}}\right]^2. \tag{83}$$

The last inequality in (83) holds because

$$\frac{4\sqrt{1-\Omega_j}\,[2+\Omega_j]}{8+\Omega_j} < 1 \text{ and } \frac{\Delta_{2j}}{\widetilde{\Delta}_{1j}} < 1. \tag{84}$$

The first inequality in (84) holds because

$$\frac{\partial\left(\sqrt{1-\Omega_j}\,\frac{4[\,2+\Omega_j\,]}{8+\Omega_j}\right)}{\partial\Omega_j} \;=\; \sqrt{1-\Omega_j}\,\frac{\partial\left(\frac{4[\,2+\Omega_j\,]}{8+\Omega_j}\right)}{\partial\Omega_j} + \frac{\partial\left(\sqrt{1-\Omega_j}\right)}{\partial\Omega_j}\,\frac{4\,[\,2+\Omega_j\,]}{8+\Omega_j}$$

$$=\; \sqrt{1-\Omega_j}\,\frac{4\,[\,8+\Omega_j\,]-4\,[\,2+\Omega_j\,]}{[\,8+\Omega_j\,]^2} - \frac{1}{2}\,\frac{1}{\sqrt{1-\Omega_j}}\,\frac{4\,[\,2+\Omega_j\,]}{8+\Omega_j}$$

$$=\; \frac{24\sqrt{1-\Omega_j}}{[\,8+\Omega_j\,]^2} - \frac{2\,[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}\;[\,8+\Omega_j\,]} \;=\; \frac{2}{8+\Omega_j}\left[\frac{12\sqrt{1-\Omega_j}}{8+\Omega_j} - \frac{[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}}\right] \;<\; 0, \tag{85}$$

and therefore, for $\Omega_j \in (0,1)$,

$$\frac{4\sqrt{1-\Omega_j}\,[\,2+\Omega_j\,]}{8+\Omega_j} \;<\; \max\frac{4\sqrt{1-\Omega_j}\,[\,2+\Omega_j\,]}{8+\Omega_j} \;=\; \frac{4\sqrt{1-0}\,[\,2+0\,]}{8+0} \;=\; 1. \tag{86}$$

The last inequality in (85) holds because

$$\frac{12\sqrt{1-\Omega_j}}{8+\Omega_j} \;<\; \frac{[\,2+\Omega_j\,]}{\sqrt{1-\Omega_j}} \;\Leftrightarrow\; 12\,[\,1-\Omega_j\,] \;<\; [\,2+\Omega_j\,]\,[\,8+\Omega_j\,]$$

$$\Leftrightarrow\; 12-12\,\Omega_j \;<\; 16+[\Omega_j]^2+10\,\Omega_j \;\Leftrightarrow\; 4+[\Omega_j]^2+22\,\Omega_j \;>\; 0.$$

The last inequality in (84) holds because

$$\widetilde{\Delta}_{1j} \;=\; \widetilde{\Delta}_{2j} \;>\; \Delta_{2j}. \tag{87}$$

The equality in (87) reflects (81) and the inequality in (87) reflects (12). ■

Lemma 10. *Suppose Condition FS holds and* $\frac{\Theta_1}{\Theta_2} \geq 1$. *Further suppose P1 makes no commitment and P2 commits to no entry. Then Sj* ($j \in \{1,2\}$) *will: (i) sell on P1 if* $\frac{\Theta_1}{\Theta_2} > \phi_j$; *and (ii) sell on P2 if* $\frac{\Theta_1}{\Theta_2} < \phi_j$.

Proof. Condition FS ensures that P1 will enter Sj's market if Sj sells on P1 ($j \in \{1,2\}$). Lemma 3 implies that Sj's profit is $\frac{\Theta_2[\,\widetilde{\Delta}_{2j}\,]^2}{16\,b_{2j}^S}$ if Sj sells on P2. Lemma 6 implies that Sj's profit is $\frac{\Theta_1}{\beta_j^S}\left[\frac{(2+\Omega_j\,)\,\Delta_{1j}}{8+\Omega_j}\right]^2$ if Sj sells on P1. (10) and (17) imply that:

$$\frac{\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2 \;\gtreqless\; \frac{\Theta_2\left[\,\widetilde{\Delta}_{2j}\,\right]^2}{16\,b_{2j}^S} \;\Leftrightarrow\; \frac{\Theta_1}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{1j}}{8+\Omega_j}\right]^2 \;\gtreqless\; \frac{\Theta_2\left[\,\widetilde{\Delta}_{2j}\,\right]^2}{16\,\beta_j^S\,[\,1-\Omega_j\,]}$$

$$\Leftrightarrow \quad \frac{\Theta_1}{\Theta_2} \gtreqless \left[\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,(\,2+\Omega_j\,)} \frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} \right]^2 \quad \Leftrightarrow \quad \frac{\Theta_1}{\Theta_2} \gtreqless \phi_j . \tag{88}$$

(86) implies that $\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}[\,2+\Omega_j\,]} > 1$. (12) and (81) imply that $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j} > \Delta_{1j}$. Therefore,

$$\phi_j \;>\; 1. \tag{89}$$

(88) and (89) imply that Sj will: (i) sell on P1 if $\frac{\Theta_1}{\Theta_2} > \phi_j$; and (ii) sell on P2 if $\frac{\Theta_1}{\Theta_2} < \phi_j$. ■

Proposition 2. *Suppose Condition FS and Assumption BC hold, and* $\frac{\Theta_1}{\Theta_2} \geq 1$. *Then in equilibrium: (i) if* $\frac{\Theta_1}{\Theta_2} > \phi_2$, *P1 makes no commitment, and both S1 and S2 sell on P1; (ii) if* $\frac{\Theta_1}{\Theta_2} \in (\,\phi_1, \phi_2\,)$, *P1 makes no commitment whereas P2 commits not to enter, and S1 sells on P1 whereas S2 sells on P2; (iii) if* $\frac{\Theta_1}{\Theta_2} \in (\,1, \phi_1\,)$, *P1 commits not to enter, and both S1 and S2 sell on P1; and (iv) if* $\frac{\Theta_1}{\Theta_2} = 1$, *both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2.*

Proof. Condition FS ensures that each platform enters each seller's market if the platform makes no commitment. Since S1 and S2 sell independent products, S1's choice of platform is independent of S2's choice of platform.

Case I. $c^P_{1j} < c^P_{2j}$.

Observe that:

$$\frac{\partial \left(\frac{\widetilde{\Delta}_{2j}}{\Delta_{1j}} \right)}{\partial c^S_j} \stackrel{s}{=} \frac{\partial \widetilde{\Delta}_{2j}}{\partial c^S_j} \Delta_{1j} - \frac{\partial \Delta_{1j}}{\partial c^S_j} \widetilde{\Delta}_{2j} \;=\; -\,b^S_j\,\Delta_{1j} + \beta^S_j\,\widetilde{\Delta}_{2j} \;=\; -\,b^S_j\,\Delta_{1j} + \beta^S_j\,\widetilde{\Delta}_{1j} \;>\; 0. \tag{90}$$

The last equality in (90) holds because $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j}$ (due to $\frac{\partial \widetilde{\Delta}_{kj}}{\partial c^P_{kj}} = 0$). The last inequality in (90) holds because $\beta^S_j > b^S_j$ (from (10)) and $\widetilde{\Delta}_{1j} > \Delta_{1j}$ (from (12)).

$c^S_1 < c^S_2$, (17), and (90) imply that:

$$\phi_2 \;>\; \phi_1. \tag{91}$$

Further observe that:

$$\frac{\partial \left(\frac{\Delta_{2j}}{\Delta_{1j}} \right)}{\partial c^S_j} \stackrel{s}{=} \frac{\partial \Delta_{2j}}{\partial c^S_j} \Delta_{1j} - \frac{\partial \Delta_{1j}}{\partial c^S_j} \Delta_{2j} \;=\; -\beta^S_j\,\Delta_{1j} + \beta^S_j\,\Delta_{2j} \;=\; \beta^S_j\,[\,\Delta_{2j} - \Delta_{1j}\,] \;=\; \beta^S_j\,\eta_j\,[\,c^P_{2j} - c^P_{1j}\,] \;>\; 0. \tag{92}$$

The inequality in (92) holds because $c^P_{1j} < c^P_{2j}$ by assumption.

$c_1^S < c_2^S$ and (92) imply that:

$$\left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 > \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2. \tag{93}$$

Because $c_{1j}^P < c_{2j}^P$, then $\frac{\Delta_{2j}}{\Delta_{1j}} > 1$ for $j \in \{1, 2\}$. Therefore, (91), (93), and Assumption BC imply that

$$\phi_2 > \phi_1 > \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 > \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 > 1 \text{ if } c_{1j}^P < c_{2j}^P. \tag{94}$$

First suppose $\frac{\Theta_1}{\Theta_2} > \phi_2$. Lemmas 7 - 10 and (94) imply that both S1 and S2 sells on P1, regardless of the platforms' commitments. Condition FS ensures P1 enters each seller's market. Therefore, Lemmas 3 and 6 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 [\widetilde{\Delta}_{1j}]^2}{8 b_j^S}$, i.e., P1 secures more profit by making no commitment than by committing not to enter. Therefore, in equilibrium, P1 makes no commitment, and both S1 and S2 sell on P1 if $\frac{\Theta_1}{\Theta_2} > \phi_2$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$. Lemmas 7 - 10 and (94) imply that S1 sells on P1, regardless of the platforms' commitments.

If P2 makes no commitment, Lemmas 8, 9 and (94) imply that S2 sells on P1. Condition FS and Lemmas 3 and 6 imply that P1 secures more profit by making no commitment than by committing not to enter in this case.

If P2 commits not to enter, Lemmas 7 and 10 imply that S2: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Lemmas 3 and 6 imply that P1's profit is: (i) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8 b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8 b_2^S}$ if P1 commits not to enter; and (ii) $\Theta_1 M_{11} - F$ if P1 makes no commitment. Therefore, P1 secures more profit by making no commitment than by committing not to enter in this case because

$$\Theta_1 M_{11} - F > \frac{\Theta_1 \left[\widetilde{\Delta}_{11}\right]^2}{8 b_1^S} + \frac{\Theta_1 \left[\widetilde{\Delta}_{12}\right]^2}{8 b_2^S} \Leftrightarrow F < \Theta_1 M_{11} - \frac{\Theta_1 \left[\widetilde{\Delta}_{11}\right]^2}{8 b_1^S} - \frac{\Theta_1 \left[\widetilde{\Delta}_{12}\right]^2}{8 b_2^S}.$$

If P1 commits not to enter, Lemmas 7 and 9 imply that S2 sells on P1, regardless of P2's commitment.

If P1 makes no commitment, Lemmas 8 and 10 imply that: (i) S2 sells on P1 if P2 makes no commitment; and (ii) S2 sells on P2 if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment.

Consequently, in equilibrium, P1 makes no commitment whereas P2 commits not to enter, and S1 sells on P1 whereas S2 sells on P2, if $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2, \phi_1 \right)$. If P2 makes no commitment, Lemmas 8, 9 and (94) imply that S1 and S2 both sell on P1, regardless of P1's commitment. Lemmas 3 and 6 imply that P1's profit is: (i) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8\, b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8\, b_2^S}$ if P1 commits not to enter; and (ii) $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment. Condition FS ensures that P1 secures more profit by making no commitment than by committing not to enter in this case.

If P2 commits not to enter, Lemmas 7, 10 and (94) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 commits not to enter, Lemmas 7, 9 and (94) imply that both S1 and S2 sell on P1, regardless of P2's commitment.

If P1 makes no commitment, Lemmas 8, 10, and **??** imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P2 makes no commitment; and (ii) sells on P2 if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case.

Consequently, in equilibrium, both P1 and P2 commit not to enter, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2, \phi_1 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2 \right)$. If P2 makes no commitment, Lemmas 8, 9, and (94) imply that S1 sells on P1 and S2: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Lemmas 3 and 6 imply that P1's profit is: (i) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8\, b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8\, b_2^S}$ if P1 commits not to enter; and (ii) $\Theta_1 M_{11} - F$ if P1 makes no commitment. Therefore, P1 secures more profit by making no commitment than by committing not to enter in this case because

$$\Theta_1 M_{11} - F \; > \; \frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S} \;\; \Leftrightarrow \;\; F \; < \; \Theta_1 M_{11} - \frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} - \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}.$$

If P2 commits not to enter, Lemmas 7, 10, and (94) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 commits not to enter, Lemmas 7, 9, and (94) imply that both S1 and S2 sell on P1, regardless of P2's commitment.

If P1 makes no commitment, Lemmas 8, 10, and (94) imply that S2 sells on P2 and S1: (i) sells on P1 if P2 makes no commitment; and (ii) sells on P2 if P2 commits not to enter. Lemmas 3 and 6 imply that P2's profit is: (i) $\frac{\Theta_2 \left[\widetilde{\Delta}_{21} \right]^2}{8\, b_1^S} + \frac{\Theta_2 \left[\widetilde{\Delta}_{22} \right]^2}{8\, b_2^S}$ if P2 commits not to enter; and (ii) $\Theta_2\, M_{22} - F$ if P2 makes no commitment. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case because

$$\Theta_2\, M_{22} - F \;<\; \frac{\Theta_2 \left[\widetilde{\Delta}_{21} \right]^2}{8\, b_1^S} + \frac{\Theta_2 \left[\widetilde{\Delta}_{22} \right]^2}{8\, b_2^S} \quad \Leftrightarrow \quad F \;>\; \Theta_2\, M_{22} - \frac{\Theta_2 \left[\widetilde{\Delta}_{21} \right]^2}{8\, b_1^S} - \frac{\Theta_2 \left[\widetilde{\Delta}_{22} \right]^2}{8\, b_2^S}.$$

Consequently, in equilibrium, both P1 and P2 commit not to enter, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} \in \left(\left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2, \left[\frac{\Delta_{22}}{\Delta_{12}} \right]^2 \right)$.

Next suppose $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2$. If P2 makes no commitment, Lemmas 8, 9, and (94) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P2 commits not to enter, Lemmas 7, 10, and (94) imply that Sj ($j \in \{1, 2\}$): (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 commits not to enter, Lemmas 7, 9, and (94) imply that both S1 and S2 sell on P1, regardless of P2's commitment.

If P1 makes no commitment, Lemmas 8, 10, and (94) imply that both S1 and S2 sell on P2, regardless of P2's commitment. Lemmas 3 and 6 imply that P2's profit is: (i) $\frac{\Theta_2 \left[\widetilde{\Delta}_{21} \right]^2}{8\, b_1^S} + \frac{\Theta_2 \left[\widetilde{\Delta}_{22} \right]^2}{8\, b_2^S}$ if P2 commits not to enter; and (ii) $\Theta_2\, M_{21} - F + \Theta_2\, M_{22} - F$ if P2 makes no commitment. Condition FS ensures that P2 secures more profit by making no commitment than by committing not to enter in this case.

Consequently, in equilibrium, P1 commits not to enter, and both S1 and S2 sell on P1, if $\frac{\Theta_1}{\Theta_2} < \left[\frac{\Delta_{21}}{\Delta_{11}} \right]^2$.

Finally, suppose $\frac{\Theta_1}{\Theta_2} = 1$. Lemma 7 implies that if both platforms commit not to enter, then Sj ($j \in \{1, 2\}$) is indifferent between selling on P1 and selling on P2. Lemma 8 implies that if platforms both make no commitment, then Sj sells on P2. Lemma 9 implies that if P1 commits not to enter and P2 makes no commitment, then Sj sells on P1. Lemma 10 implies that if P1 makes no commitment and P2 commits to no entry, then Sj sells on P2.

If P2 makes no commitment, Sj: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P2 commits not to enter, Sj: (i) is indifferent between selling on P1 and selling on P2 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 commits not to enter, Sj: (i) is indifferent between selling on P1 and selling on P2 if P2 commits not to enter; and (ii) sells on P1 if P2 makes no commitment. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case.

Consequently, in equilibrium, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2, if $\frac{\Theta_1}{\Theta_2} = 1$.

<u>Case II</u>. $c^P_{1j} > c^P_{2j}$.

Because $c^P_{1j} > c^P_{2j}$, then for $j \in \{1, 2\}$,

$$\frac{\Delta_{2j}}{\Delta_{1j}} < 1. \tag{95}$$

(86) implies that $\frac{8+\Omega_j}{4\sqrt{1-\Omega_j}\,[2+\Omega_j]} > 1$. $\frac{\partial \widetilde{\Delta}_{kj}}{\partial c^P_{kj}} = 0$ implies that $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j}$. (12) implies that $\widetilde{\Delta}_{1j} > \Delta_{1j}$. Therefore, $\widetilde{\Delta}_{2j} = \widetilde{\Delta}_{1j} > \Delta_{1j}$. (17) implies that for $j \in \{1, 2\}$,

$$\phi_j > 1. \tag{96}$$

Therefore, (91), (93), (95), and (96) imply that

$$\phi_2 > \phi_1 > 1 > \left[\frac{\Delta_{22}}{\Delta_{12}}\right]^2 > \left[\frac{\Delta_{21}}{\Delta_{11}}\right]^2 \text{ if } c^P_{1j} > c^P_{2j}. \tag{97}$$

First suppose $\frac{\Theta_1}{\Theta_2} > \phi_2$. Lemmas 7 - 10 and (97) imply that both S1 and S2 sell on P1, regardless of the platforms' commitments. Condition FS ensures P1 enters each seller's market. Therefore, Lemmas 3 and 6 imply that: (i) P1's profit is $\Theta_1 M_{11} - F + \Theta_1 M_{12} - F$ if P1 makes no commitment; and (ii) P1's profit is $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8\, b^S_1} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8\, b^S_2}$ if P1 commits not to enter. Condition FS ensures that $\Theta_1 M_{1j} - F > \frac{\Theta_1 [\widetilde{\Delta}_{1j}]^2}{8\, b^S_j}$, i.e., P1 secures more profit by making no commitment than by committing to no entry. Therefore, in equilibrium, P1 makes no commitment, and both S1 and S2 sell on P1 if $\frac{\Theta_1}{\Theta_2} > \phi_2$.

Next suppose $\frac{\Theta_1}{\Theta_2} \in (\phi_1, \phi_2)$. Lemmas 7 - 10 and (97) imply that S1 sells on P1, regardless of the platforms' commitments.

If P2 makes no commitment, Lemmas 8, 9 imply that S2 sells on P1. Condition FS and Lemmas 3 and 6 imply that P1 secures more profit by making no commitment than by committing not to enter in this case.

If P2 commits not to enter, Lemmas 7 and 10 imply that S2: (i) sells on P1 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Lemmas 3 and 6 imply that P1's profit is: (i) $\frac{\Theta_1 [\widetilde{\Delta}_{11}]^2}{8\, b_1^S} + \frac{\Theta_1 [\widetilde{\Delta}_{12}]^2}{8\, b_2^S}$ if P1 commits not to enter; and (ii) $\Theta_1 M_{11} - F$ if P1 makes no commitment. Therefore, Therefore, P1 secures more profit by making no commitment than by committing not to enter in this case because

$$\Theta_1 M_{11} - F \;>\; \frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} + \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S} \;\Leftrightarrow\; F \;<\; \Theta_1 M_{11} - \frac{\Theta_1 \left[\widetilde{\Delta}_{11} \right]^2}{8\, b_1^S} - \frac{\Theta_1 \left[\widetilde{\Delta}_{12} \right]^2}{8\, b_2^S}.$$

If P1 commits not to enter, Lemmas 7 and 9 imply that S2 sells on P1, regardless of P2's commitment.

If P1 makes no commitment, Lemmas 8 and 10 imply that: (i) S2 sells on P1 if P2 makes no commitment; and (ii) S2 sells P2 if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment.

Consequently, in equilibrium, P1 makes no commitment while P2 commits not to enter, and S1 sells on P1 while S2 sells on P2, if $\frac{\Theta_1}{\Theta_2} \in (\, \phi_1, \phi_2\,)$.

Next suppose $\frac{\Theta_1}{\Theta_2} < \phi_1$. If P2 makes no commitment, Lemmas 8 and 9 imply that both S1 and S2 sell on P1, regardless of P1's commitment. Condition FS and Lemmas 3 and 6 imply that P1 secures more profit by making no commitment than by committing not to enter in this case.

If P2 commits not to enter, Lemmas 7 and 10 imply that both S1 and S2 sell: (i) on P1 if P1 commits not to enter; and (ii) on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 makes no commitment, Lemmas 8 and 10 imply that both S1 and S2 sell: (i) on P1 if P2 makes no commitment; and (ii) on P2 if P2 commits not to enter. Therefore, P2 secures more profit by committing not to enter than by making no commitment.

If P1 commits not to enter, Lemmas 7 and 9 imply that both S1 and S2 sell on P1, regardless of P2's commitment.

Therefore, in equilibrium, both P1 and P2 commit not to enter and both S1 and S2 sell on P1 if $\frac{\Theta_1}{\Theta_2} < \phi_1$.

Finally, suppose $\frac{\Theta_1}{\Theta_2} = 1$. Lemma 7 implies that if both platforms commit not to enter, then Sj ($j \in \{1, 2\}$) is indifferent between selling on P1 and selling on P2. Lemma 8 implies that if platforms both make no commitment, then Sj sells on P1. Lemma 9 implies that if P1 commits not to enter and P2 makes no commitment, then Sj sells on P1. Lemma 10 implies that if P1 makes no commitment and P2 commits to no entry, then Sj sells on P2.

If P2 makes no commitment, Sj sells on P1, regardless of P1's commitment.

If P2 commits not to enter, Sj: (i) is indifferent between selling on P1 and selling on P2 if P1 commits not to enter; and (ii) sells on P2 if P1 makes no commitment. Therefore, P1 secures more profit by committing not to enter than by making no commitment in this case.

If P1 commits not to enter, Sj: (i) is indifferent between selling on P1 and selling on P2 if P2 commits not to enter; and (ii) sells on P1 if P2 makes no commitment. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case.

If P1 makes no commitment, Sj: sells on P1 if P2 makes no commitment; and (ii) sells on P2 if P2 commits to no entry. Therefore, P2 secures more profit by committing not to enter than by making no commitment in this case.

Consequently, in equilibrium, both platforms commit not to enter, and each seller is indifferent between selling on P1 and selling on P2, if $\frac{\Theta_1}{\Theta_2} = 1$. ■

Proposition 3. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform* $\widetilde{P}$ *that is a weaker seller than P (i.e.,* $\frac{\widetilde{c}_j^P}{c_j^P} > 1$*). Then increased platform competition reduces platform-seller competition in the sense that at least one seller faces no competition or reduced competition and no seller faces increased competition in the presence of platform competition.*

Proof. Proposition 1 implies that Sj ($j \in \{1, 2\}$) competes against P under MP.

First suppose P faces a competing platform $\widetilde{\text{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC, where $\widetilde{\Theta}$ denotes $\widetilde{\text{P}}$'s platform strength.

Case I. $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.

Proposition 2 implies that each seller competes against $\widetilde{\text{P}}$ under PC. Because $\widetilde{\text{P}}$ is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$), increased platform competition reduces platform-seller competition in the sense that each seller faces reduced competition under PC, whereas each seller competes against P under MP.[3]

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in (\phi_1, \phi_2)$.

Proposition 2 implies that S1 competes against $\widetilde{\text{P}}$ whereas S2 faces no competition under PC. Because $\widetilde{\text{P}}$ is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$), increased platform competition reduces platform-seller competition in the sense that S1 faces reduced competition and S2 faces no competition under PC, whereas each seller competes against P under MP.

Case III. $\frac{\widetilde{\Theta}}{\Theta} \in (1, \phi_1)$.

[3] $\widetilde{c}_j^P$ denotes $\widetilde{\text{P}}$'s cost of imitating Sj's product, and c_j^P denotes P's cost of imitating Sj's product.

Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Next suppose P faces a symmetric competing platform $\widetilde{\text{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Finally, suppose P faces a competing platform $\widetilde{\text{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1 \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (1, \phi_1)$. Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1} \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\phi_1, \phi_2)$. Proposition 2 implies that S1 competes against P whereas S2 faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that S2 faces no competition and S1 faces unchanged competition under PC, whereas each seller competes against P under MP. ■

Proposition 4. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform $\widetilde{P}$ that is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$). Then $\pi_1 \geq \pi_1^M$ and $\pi_2 > \pi_2^M$, where the first inequality holds strictly unless $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1} \right)$.*

Proof. Proposition 1 indicates that Sj's ($j \in \{1, 2\}$) equilibrium profit under MP is

$$\pi_j^M = \frac{\Theta}{\beta_j^S} \left[\frac{(2 + \Omega_j)\, \Delta_{Pj}}{8 + \Omega_j} \right]^2 . \tag{98}$$

First suppose P faces a competing platform $\widetilde{\text{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.

Proposition 2 implies that each seller competes against $\widetilde{\text{P}}$. Lemma 6 implies that Sj's equilibrium profit is

$$\pi_j = \frac{\widetilde{\Theta}}{\beta_j^S} \left[\frac{(2 + \Omega_j)\, \Delta_{\widetilde{P}j}}{8 + \Omega_j} \right]^2 . \tag{99}$$

(98) and (99) imply that:

$$\pi_j \;>\; \pi_j^M \quad \Leftrightarrow \quad \widetilde{\Theta}\left[\Delta_{\widetilde{P}j}\right]^2 \;>\; \Theta\left[\Delta_{Pj}\right]^2 . \tag{100}$$

The last inequality in (100) holds because $\frac{\widetilde{\Theta}}{\Theta} > \phi_2 > 1$ and $\Delta_{\widetilde{P}j} > \Delta_{Pj}$ (due to $\frac{\widetilde{c}_j^P}{c_j^P} > 1$).

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in (\,\phi_1, \phi_2\,)$.

Proposition 2 implies that S1 competes against $\widetilde{\mathrm{P}}$ whereas S2 faces no competition on P. Lemmas 3 and 6 imply that S1's equilibrium profit is $\pi_1 = \frac{\widetilde{\Theta}}{\beta_1^S}\left[\frac{(2+\Omega_1)\,\Delta_{\widetilde{P}1}}{8+\Omega_1}\right]^2$ and S2's equilibrium profit is $\pi_2 = \frac{\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{16\, b_2^S}$. Therefore, (100) implies $\pi_1 > \pi_1^M$. (10) and (98) imply that:

$$\pi_2 \;>\; \pi_2^M \quad \Leftrightarrow \quad \frac{\Theta\left[\,\widetilde{\Delta}_{P2}\,\right]^2}{16\,\beta_2^S\,[\,1-\Omega_2\,]} \;>\; \frac{\Theta}{\beta_2^S}\left[\frac{(\,2+\Omega_2\,)\;\Delta_{P2}}{8+\Omega_2}\right]^2$$

$$\Leftrightarrow \quad \left[\frac{\widetilde{\Delta}_{P2}}{\Delta_{P2}}\right]^2 \;>\; \left[\frac{4\sqrt{1-\Omega_2}\,(\,2+\Omega_2\,)}{8+\Omega_2}\right]^2 . \tag{101}$$

(101) holds because $\frac{\widetilde{\Delta}_{P2}}{\Delta_{P2}} > 1$ and $\left[\frac{4\sqrt{1-\Omega_2}(\,2+\Omega_2\,)}{8+\Omega_2}\right]^2 < 1$ (from (86)).

<u>Case III</u>. $\frac{\widetilde{\Theta}}{\Theta} \in (\,1, \phi_1\,)$.

Proposition 2 implies that each seller sells on $\widetilde{\mathrm{P}}$ and faces no competition. Lemma 3 implies that Sj's equilibrium profit is $\pi_j = \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{16\, b_j^S}$. (98) implies that:

$$\pi_j \;>\; \pi_j^M \quad \Leftrightarrow \quad \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{16\, b_j^S} \;>\; \frac{\Theta}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{Pj}}{8+\Omega_j}\right]^2 . \tag{102}$$

(102) holds because

$$\frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{16\, b_j^S} \;>\; \frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{16\, b_j^S} \;>\; \frac{\Theta}{\beta_j^S}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{Pj}}{8+\Omega_j}\right]^2 . \tag{103}$$

The first inequality in (103) reflects $\widetilde{\Theta} > \Theta$ and $\widetilde{\Delta}_{\widetilde{P}j} = \widetilde{\Delta}_{Pj}$ (due to $\frac{\partial \widetilde{\Delta}_{kj}}{\partial c_{kj}^P} = 0$). The second inequality in (103) reflects (101).

Next suppose P faces a symmetric competing platform $\widetilde{\mathrm{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$). Proposition 2 implies that each seller faces no competition and is indifferent between selling on P and selling on $\widetilde{\text{P}}$ under PC. Lemma 3 implies that Sj's equilibrium profit is

$$\pi_j \;=\; \frac{1}{2}\frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{16\,b_j^S} + \frac{1}{2}\frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{16\,b_j^S} \;>\; \frac{1}{2}\frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2 + \Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{16\,b_j^S} \;=\; \frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{16\,b_j^S}. \tag{104}$$

The inequality in (104) reflects $\widetilde{\Theta} > \Theta$ and $\widetilde{\Delta}_{\widetilde{P}j} = \widetilde{\Delta}_{Pj}$ (due to $\frac{\partial \widetilde{\Delta}_{kj}}{\partial c_{kj}^P} = 0$). (98), (103), and (104) imply that $\pi_j > \pi_j^M$.

Finally, suppose P faces a competing platform $\widetilde{\text{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,1, \phi_1\,)$. Proposition 2 implies that each seller sells on P and faces no competition under PC. Lemma 3 implies that Sj's equilibrium profit is $\pi_j = \frac{\Theta\left[\widetilde{\Delta}_{Pj}\right]^2}{16\,b_j^S}$. (98) and (103) imply that $\pi_j > \pi_j^M$.

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1}\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,\phi_1, \phi_2\,)$. Proposition 2 implies that S1 competes against P whereas S2 sells on $\widetilde{\text{P}}$ and faces no competition under PC. Lemmas 3 and 6 imply that S1's equilibrium profit is $\pi_1 = \frac{\Theta}{\beta_1^S}\left[\frac{(2+\Omega_1)\,\Delta_{P1}}{8+\Omega_1}\right]^2$ and S2's equilibrium profit is $\pi_2 = \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{16\,b_2^S}$. (98) implies that $\pi_1 = \pi_1^M$. (102) implies that $\pi_2 > \pi_2^M$. ■

Proposition 5. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform $\widetilde{P}$ that is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$). Then $CS < CS^M$ unless $\widetilde{P}$'s relative platform strength is sufficiently pronounced.*

Proof. (1) implies that if Sj sells on Pk and competes against Pk ($j, k \in \{1,2\}$), then:

$$p_{kj}^S \;=\; \frac{q_{kj}^P}{\eta_j} - \frac{\theta_j\,\alpha_j}{\eta_j} + \frac{\beta_j^P\,p_{kj}^P}{\eta_j}. \tag{105}$$

(2) and (105) imply that:

$$q_{kj}^S \;=\; \alpha_j - \frac{\beta_j^S\,q_{kj}^P}{\eta_j} + \frac{\beta_j^S\,\theta_j\,\alpha_j}{\eta_j} - \frac{\beta_j^S\,\beta_j^P\,p_{kj}^P}{\eta_j} + \eta_j\,p_{kj}^P$$

$$\Leftrightarrow\;\; q_{kj}^S \;=\; \alpha_j - \frac{\beta_j^S\,q_{kj}^P}{\eta_j} + \frac{\beta_j^S\,\theta_j\,\alpha_j}{\eta_j} + \frac{\left[\eta_j\right]^2 - \beta_j^S\,\beta_j^P}{\eta_j}\,p_{kj}^P$$

$$\Leftrightarrow \quad \eta_j\, q^S_{kj} \;=\; \eta_j\, \alpha_j - \beta^S_j\, q^P_{kj} + \beta^S_j\, \theta_j\, \alpha_j + \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big]\; p^P_{kj}$$

$$\Leftrightarrow \quad p^P_{kj} \;=\; \frac{\eta_j\, q^S_{kj} + \beta^S_j\, q^P_{kj} - \alpha_j \left[\, \eta_j + \beta^S_j\, \theta_j \,\right]}{\left[\eta_j\right]^2 - \beta^S_j\, \beta^P_j}. \tag{106}$$

(105) and (106) imply that:

$$p^S_{kj} \;=\; \frac{q^P_{kj}}{\eta_j} - \frac{\theta_j\, \alpha_j}{\eta_j} + \frac{\beta^P_j}{\eta_j}\, \frac{\eta_j\, q^S_{kj} + \beta^S_j\, q^P_{kj} - \alpha_j \left[\, \eta_j + \beta^S_j\, \theta_j \,\right]}{\left[\eta_j\right]^2 - \beta^S_j\, \beta^P_j}$$

$$= \; \frac{q^P_{kj} \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big] - \theta_j\, \alpha_j \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big] + \beta^P_j\, \eta_j\, q^S_{kj} + \beta^P_j\, \beta^S_j\, q^P_{kj} - \beta^P_j\, \alpha_j \left[\, \eta_j + \beta^S_j\, \theta_j \,\right]}{\eta_j \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big]}$$

$$= \; \frac{q^P_{kj} \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j + \beta^P_j\, \beta^S_j \Big] - \alpha_j \Big[\theta_j\, \left(\eta_j\right)^2 - \theta_j\, \beta^S_j\, \beta^P_j \Big] + \beta^P_j\, \eta_j\, q^S_{kj} - \alpha_j \left[\, \beta^P_j\, \eta_j + \beta^P_j\, \beta^S_j\, \theta_j \,\right]}{\eta_j \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big]}$$

$$= \; \frac{\left[\eta_j\right]^2\, q^P_{kj} - \alpha_j \Big[\theta_j\, \left(\eta_j\right)^2 - \theta_j\, \beta^S_j\, \beta^P_j + \beta^P_j\, \eta_j + \beta^P_j\, \beta^S_j\, \theta_j \Big] + \beta^P_j\, \eta_j\, q^S_{kj}}{\eta_j \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big]}$$

$$= \; \frac{\left[\eta_j\right]^2\, q^P_{kj} - \eta_j\, \alpha_j \left[\, \theta_j\, \eta_j + \beta^P_j \,\right] + \beta^P_j\, \eta_j\, q^S_{kj}}{\eta_j \Big[\left(\eta_j\right)^2 - \beta^S_j\, \beta^P_j \Big]} \;=\; \frac{\eta_j\, q^P_{kj} + \beta^P_j\, q^S_{kj} - \alpha_j \left[\, \theta_j\, \eta_j + \beta^P_j \,\right]}{\left[\eta_j\right]^2 - \beta^S_j\, \beta^P_j}. \tag{107}$$

(106) and (107) imply that if S1 competes against Pk and S2 competes against Pi ($k, i \in \{1, 2\}$), then consumer surplus is:

$$CS \;=\; U\left(Q^{*P}_{k1}, Q^{*P}_{i2}, Q^{*S}_{k1}, Q^{*S}_{i2}\right) - p^{*P}_{k1}\, Q^{*P}_{k1} - p^{*P}_{i2}\, Q^{*P}_{i2} - p^{*S}_{k1}\, Q^{*S}_{k1} - p^{*S}_{i2}\, Q^{*S}_{i2} \tag{108}$$

$$\begin{aligned}
&= \frac{\frac{1}{2}\, \eta_1\, \frac{Q^{*S}_{k1}\, Q^{*P}_{k1}}{\Theta_k} + \frac{1}{2}\, \beta^S_1 \frac{\left[Q^{*P}_{k1}\right]^2}{\Theta_k} - \alpha_1 \left[\, \eta_1 + \beta^S_1\, \theta_1 \,\right]\; Q^{*P}_{k1}}{\left[\eta_1\right]^2 - \beta^S_1\, \beta^P_1} \\
&\quad + \frac{\frac{1}{2}\, \eta_2\, \frac{Q^{*S}_{i2}\, Q^{*P}_{i2}}{\Theta_i} + \frac{1}{2}\, \beta^S_2 \frac{\left[Q^{*P}_{i2}\right]^2}{\Theta_i} - \alpha_2 \left[\, \eta_2 + \beta^S_2\, \theta_2 \,\right]\; Q^{*P}_{i2}}{\left[\eta_2\right]^2 - \beta^S_2\, \beta^P_2} \\
&\quad + \frac{\frac{1}{2}\, \eta_1\, \frac{Q^{*S}_{k1}\, Q^{*P}_{k1}}{\Theta_k} + \frac{1}{2}\, \beta^P_1 \frac{\left[Q^{*S}_{k1}\right]^2}{\Theta_k} - \alpha_1 \left[\, \theta_1\, \eta_1 + \beta^P_1 \,\right]\; Q^{*S}_{k1}}{\left[\eta_1\right]^2 - \beta^S_1\, \beta^P_1} \\
&\quad + \frac{\frac{1}{2}\, \eta_2\, \frac{Q^{*P}_{i2}\, Q^{*S}_{i2}}{\Theta_i} + \frac{1}{2}\, \beta^P_2 \frac{\left[Q^{*S}_{i2}\right]^2}{\Theta_i} - \alpha_2 \left[\, \theta_2\, \eta_2 + \beta^P_2 \,\right]\; Q^{*S}_{i2}}{\left[\eta_2\right]^2 - \beta^S_2\, \beta^P_2} \\
&\quad - p^{*P}_{k1}\, Q^{*P}_{k1} - p^{*P}_{i2}\, Q^{*P}_{i2} - p^{*S}_{k1}\, Q^{*S}_{k1} - p^{*S}_{i2}\, Q^{*S}_{i2}
\end{aligned} \tag{109}$$

$$
\begin{aligned}
= \;& \frac{\eta_1 \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^S \frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} + \frac{1}{2}\,\beta_1^P \frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k}}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} + \frac{\eta_2 \frac{Q_{i2}^{*S}\, Q_{i2}^{*P}}{\Theta_i} + \frac{1}{2}\,\beta_2^S \frac{\left[Q_{i2}^{*P}\right]^2}{\Theta_i} + \frac{1}{2}\,\beta_2^P \frac{\left[Q_{i2}^{*S}\right]^2}{\Theta_i}}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
& - \left[\frac{\alpha_1\left(\eta_1 + \beta_1^S\,\theta_1\right)}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P} + p_{k1}^{*P}\right] Q_{k1}^{*P} - \left[\frac{\alpha_2\left(\eta_2 + \beta_2^S\,\theta_2\right)}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P} + p_{i2}^{*P}\right] Q_{i2}^{*P} \\
& - \left[\frac{\alpha_1\left(\theta_1\,\eta_1 + \beta_1^P\right)}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P} + p_{k1}^{*S}\right] Q_{k1}^{*S} - \left[\frac{\alpha_2\left(\theta_2\,\eta_2 + \beta_2^P\right)}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P} + p_{i2}^{*S}\right] Q_{i2}^{*S} \\
= \;& \frac{\eta_1 \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^S \frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} + \frac{1}{2}\,\beta_1^P \frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k}}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} + \frac{\eta_2 \frac{Q_{i2}^{*S}\, Q_{i2}^{*P}}{\Theta_i} + \frac{1}{2}\,\beta_2^S \frac{\left[Q_{i2}^{*P}\right]^2}{\Theta_i} + \frac{1}{2}\,\beta_2^P \frac{\left[Q_{i2}^{*S}\right]^2}{\Theta_i}}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
& - \left[\frac{\eta_1\, q_{k1}^{*S} + \beta_1^S\, q_{k1}^{*P}}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P}\right] Q_{k1}^{*P} - \left[\frac{\eta_2\, q_{i2}^{*S} + \beta_2^S\, q_{i2}^{*P}}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P}\right] Q_{i2}^{*P} \\
& - \left[\frac{\eta_1\, q_{k1}^{*P} + \beta_1^P\, q_{k1}^{*S}}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P}\right] Q_{k1}^{*S} - \left[\frac{\eta_2\, q_{i2}^{*P} + \beta_2^P\, q_{i2}^{*S}}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P}\right] Q_{i2}^{*S} \qquad (110)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\eta_1\,\Theta_k\, q_{k1}^{*P}\, q_{k1}^{*S} + \frac{1}{2}\,\beta_1^S\,\Theta_k\left[q_{k1}^{*P}\right]^2 + \frac{1}{2}\,\beta_1^P\,\Theta_k\left[q_{k1}^{*S}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} + \frac{\eta_2\,\Theta_i\, q_{i2}^{*P}\, q_{i2}^{*S} + \frac{1}{2}\,\beta_2^S\,\Theta_i\left[q_{i2}^{*P}\right]^2 + \frac{1}{2}\,\beta_2^P\,\Theta_i\left[q_{i2}^{*S}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
& - \left[\frac{\eta_1\, q_{k1}^{*S} + \beta_1^S\, q_{k1}^{*P}}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P}\right] \Theta_k\, q_{k1}^{*P} - \left[\frac{\eta_2\, q_{i2}^{*S} + \beta_2^S\, q_{i2}^{*P}}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P}\right] \Theta_i\, q_{i2}^{*P} \\
& - \left[\frac{\eta_1\, q_{k1}^{*P} + \beta_1^P\, q_{k1}^{*S}}{\left(\eta_1\right)^2 - \beta_1^S\,\beta_1^P}\right] \Theta_k\, q_{k1}^{*S} - \left[\frac{\eta_2\, q_{i2}^{*P} + \beta_2^P\, q_{i2}^{*S}}{\left(\eta_2\right)^2 - \beta_2^S\,\beta_2^P}\right] \Theta_i\, q_{i2}^{*S} \qquad (111)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\eta_1\,\Theta_k\, q_{k1}^{*P}\, q_{k1}^{*S} + \frac{1}{2}\,\beta_1^S\,\Theta_k\left[q_{k1}^{*P}\right]^2 + \frac{1}{2}\,\beta_1^P\,\Theta_k\left[q_{k1}^{*S}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} + \frac{\eta_2\,\Theta_i\, q_{i2}^{*P}\, q_{i2}^{*S} + \frac{1}{2}\,\beta_2^S\,\Theta_i\left[q_{i2}^{*P}\right]^2 + \frac{1}{2}\,\beta_2^P\,\Theta_i\left[q_{i2}^{*S}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
& - \Theta_k\,\frac{\eta_1\, q_{k1}^{*S}\, q_{k1}^{*P} + \beta_1^S\left[q_{k1}^{*P}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} - \Theta_i\,\frac{\eta_2\, q_{i2}^{*S}\, q_{i2}^{*P} + \beta_2^S\left[q_{i2}^{*P}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
& - \Theta_k\,\frac{\eta_1\, q_{k1}^{*P}\, q_{k1}^{*S} + \beta_1^P\left[q_{k1}^{*S}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} - \Theta_i\,\frac{\eta_2\, q_{i2}^{*P}\, q_{i2}^{*S} + \beta_2^P\left[q_{i2}^{*S}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
= \;& \Theta_k\,\frac{\eta_1\, q_{k1}^{*P}\, q_{k1}^{*S} + \frac{1}{2}\,\beta_1^S\left[q_{k1}^{*P}\right]^2 + \frac{1}{2}\,\beta_1^P\left[q_{k1}^{*S}\right]^2 - \eta_1\, q_{k1}^{*S}\, q_{k1}^{*P} - \beta_1^S\left[q_{k1}^{*P}\right]^2 - \eta_1\, q_{k1}^{*P}\, q_{k1}^{*S} - \beta_1^P\left[q_{k1}^{*S}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} \\
+ \;& \Theta_i\,\frac{\eta_2\, q_{i2}^{*P}\, q_{i2}^{*S} + \frac{1}{2}\,\beta_2^S\left[q_{i2}^{*P}\right]^2 + \frac{1}{2}\,\beta_2^P\left[q_{i2}^{*S}\right]^2 - \eta_2\, q_{i2}^{*S}\, q_{i2}^{*P} - \beta_2^S\left[q_{i2}^{*P}\right]^2 - \eta_2\, q_{i2}^{*P}\, q_{i2}^{*S} - \beta_2^P\left[q_{i2}^{*S}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P} \\
= \;& \Theta_k\,\frac{-\eta_1\, q_{k1}^{*P}\, q_{k1}^{*S} - \frac{1}{2}\,\beta_1^S\left[q_{k1}^{*P}\right]^2 - \frac{1}{2}\,\beta_1^P\left[q_{k1}^{*S}\right]^2}{\left[\eta_1\right]^2 - \beta_1^S\,\beta_1^P} + \Theta_i\,\frac{-\eta_2\, q_{i2}^{*P}\, q_{i2}^{*S} - \frac{1}{2}\,\beta_2^S\left[q_{i2}^{*P}\right]^2 - \frac{1}{2}\,\beta_2^P\left[q_{i2}^{*S}\right]^2}{\left[\eta_2\right]^2 - \beta_2^S\,\beta_2^P}
\end{aligned}
$$

$$= \Theta_k \frac{\eta_1 q^{*P}_{k1} q^{*S}_{k1} + \frac{1}{2}\beta^S_1 \left[q^{*P}_{k1}\right]^2 + \frac{1}{2}\beta^P_1 \left[q^{*S}_{k1}\right]^2}{\beta^S_1 \beta^P_1 - [\eta_1]^2} + \Theta_i \frac{\eta_2 q^{*P}_{i2} q^{*S}_{i2} + \frac{1}{2}\beta^S_2 \left[q^{*P}_{i2}\right]^2 + \frac{1}{2}\beta^P_2 \left[q^{*S}_{i2}\right]^2}{\beta^S_2 \beta^P_2 - [\eta_2]^2}. \tag{112}$$

(110) reflects (106) and (107). (111) reflects (11) and $\Theta_k \equiv B_k [1 - f_i]$.

Lemma 6 implies that if Sj competes against Pk, then:

$$q^{*S}_{kj} = \frac{Q^{*S}_{kj}}{\Theta_k} = \frac{[2 + \Omega_j]\,\Delta_{kj}}{8 + \Omega_j} \text{ and } q^{*P}_{kj} = \frac{Q^{*P}_{kj}}{\Theta_k} = \frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j [2 + \Omega_j]\,\Delta_{kj}}{2\beta^S_j [8 + \Omega_j]}. \tag{113}$$

(112) and (113) imply that if S1 competes against Pk and S2 competes against Pi ($k, i \in \{1, 2\}$), then consumer surplus is:

$$CS = \Theta_k \varsigma_{k1} + \Theta_i \varsigma_{i2}, \tag{114}$$

where for $j \in \{1, 2\}$,

$$\varsigma_{kj} \equiv \frac{1}{\beta^S_j \beta^P_j [1 - \Omega_j]} \left\{ \frac{\eta_j [2 + \Omega_j]\,\Delta_{kj}}{8 + \Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j (2 + \Omega_j)\,\Delta_{kj}}{2\beta^S_j (8 + \Omega_j)} \right] \right.$$
$$\left. + \frac{\beta^S_j}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j (2 + \Omega_j)\,\Delta_{kj}}{2\beta^S_j (8 + \Omega_j)} \right]^2 + \frac{\beta^P_j}{2} \left[\frac{(2 + \Omega_j)\,\Delta_{kj}}{8 + \Omega_j} \right]^2 \right\}. \tag{115}$$

Observe that:

$$\frac{\partial \left(\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j (2+\Omega_j)\,\Delta_{kj}}{2\beta^S_j (8+\Omega_j)} \right)}{\partial c^P_{kj}} = \frac{1}{2}\frac{\partial \overline{\Delta}_{kj}}{\partial c^P_{kj}} + \frac{\eta_j [2 + \Omega_j]}{2\beta^S_j [8 + \Omega_j]} \frac{\partial \Delta_{kj}}{\partial c^P_{kj}} = -\frac{\beta^P_j}{2} + \frac{\eta_j [2 + \Omega_j]}{2\beta^S_j [8 + \Omega_j]} \eta_j$$
$$= -\frac{\beta^P_j}{2} + \frac{[\eta_j]^2 [2 + \Omega_j]}{2\beta^S_j [8 + \Omega_j]} = -\frac{\beta^P_j}{2} + \frac{\beta^P_j \Omega_j [2 + \Omega_j]}{2 [8 + \Omega_j]} = \frac{\beta^P_j}{2} \left[\frac{\Omega_j (2 + \Omega_j)}{8 + \Omega_j} - 1 \right]$$
$$= \frac{\beta^P_j}{2} \left[\frac{2\,\Omega_j + (\Omega_j)^2 - 8 - \Omega_j}{8 + \Omega_{kj}} \right] = \frac{\beta^P_j}{2} \left[\frac{\Omega_j + (\Omega_j)^2 - 8}{8 + \Omega_j} \right] < 0; \tag{116}$$

$$\frac{\partial \left(\frac{(2+\Omega_j)\,\Delta_{kj}}{8+\Omega_j} \right)}{\partial c^P_{kj}} = \frac{2 + \Omega_j}{8 + \Omega_j} \frac{\partial \Delta_{kj}}{\partial c^P_{kj}} = \frac{\eta_j [2 + \Omega_j]}{8 + \Omega_j} > 0; \tag{117}$$

$$\frac{\partial \widetilde{\Delta}_{kj}}{\partial c^P_{kj}} = \frac{\partial \left(\Delta_{kj} + \frac{\eta_j}{\beta^P_j} \overline{\Delta}_{kj} \right)}{\partial c^P_{kj}} = \frac{\partial \Delta_{kj}}{\partial c^P_{kj}} + \frac{\eta_j}{\beta^P_j} \frac{\partial \overline{\Delta}_{kj}}{\partial c^P_{kj}} = \eta_j - \frac{\eta_j}{\beta^P_j} \beta^P_j = 0. \tag{118}$$

The inequality in (116) holds because $\Omega_j \in (0, 1)$.

(115) and (116) - (118) imply that:

$$\frac{\partial \varsigma_{kj}}{\partial c^P_{kj}} \overset{s}{=} \frac{\eta_j\,[\,2+\Omega_j\,]}{8+\Omega_j}\left\{\frac{\partial \Delta_{kj}}{\partial c^P_{kj}}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]+\Delta_{kj}\frac{\partial\left(\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j(\,8+\Omega_j\,)}\right)}{\partial c^P_{kj}}\right\} \tag{119}$$

$$+\ \frac{\beta^S_j}{2}2\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\frac{\partial\left(\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j(\,8+\Omega_j\,)}\right)}{\partial c^P_{kj}}$$

$$+\ \frac{\beta^P_j}{2}2\left[\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right]\frac{\partial\left(\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right)}{\partial c^P_{kj}}$$

$$= \frac{\eta_j\,[\,2+\Omega_j\,]}{8+\Omega_j}\left\{\eta_j\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]+\Delta_{kj}\,\frac{\beta^P_j}{2}\left[\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}\right]\right\}$$

$$+\ \beta^S_j\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\frac{\beta^P_j}{2}\left[\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}\right]$$

$$+\ \beta^P_j\left[\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right]\frac{\eta_j\,[\,2+\Omega_j\,]}{8+\Omega_j}$$

$$= \frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]}{8+\Omega_j}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]+\frac{\beta^P_j\,\eta_j\,\Delta_{kj}\,[\,2+\Omega_j\,]}{2\,[\,8+\Omega_j\,]}\left[\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}\right]$$

$$+\ \frac{\beta^S_j\,\beta^P_j}{2}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\left[\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}\right]+\frac{\beta^P_j\,\eta_j\,\Delta_{kj}\,[\,2+\Omega_j\,]^2}{[\,8+\Omega_j\,]^2}$$

$$= \left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\left[\frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)}{8+\Omega_j}+\frac{\beta^S_j\,\beta^P_j}{2}\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}\right]$$

$$+\ \frac{\beta^P_j\,\eta_j\,\Delta_{kj}\,[\,2+\Omega_j\,]}{2\,[\,8+\Omega_j\,]}\left[\frac{\Omega_j+(\Omega_j)^2\,-8}{8+\Omega_j}+\frac{2\,(\,2+\Omega_j\,)}{8+\Omega_j}\right]$$

$$= \left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\left[\frac{2\left(\eta_j\right)^2(\,2+\Omega_j\,)+\beta^S_j\,\beta^P_j\left(\,\Omega_j+(\Omega_j)^2\,-8\,\right)}{2\,(\,8+\Omega_j\,)}\right]$$

$$+\ \frac{\beta^P_j\,\eta_j\,\Delta_{kj}\,[\,2+\Omega_j\,]}{2\,[\,8+\Omega_j\,]}\left[\frac{\Omega_j+(\Omega_j)^2\,-8+4+2\,\Omega_j}{8+\Omega_j}\right]$$

$$= \beta^S_j\,\beta^P_j\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\left[\frac{2\frac{\left(\eta_j\right)^2}{\beta^S_j\,\beta^P_j}(\,2+\Omega_j\,)+\Omega_j+(\Omega_j)^2-8}{2\,(\,8+\Omega_j\,)}\right]$$

$$+ \frac{\beta_j^P \, \eta_j \, \Delta_{kj} \, [\, 2 + \Omega_j \,]}{2 \, [\, 8 + \Omega_j \,]} \left[\frac{3 \, \Omega_j + (\Omega_j)^2 - 4}{8 + \Omega_j} \right]$$

$$= \beta_j^S \, \beta_j^P \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2\beta_j^S \, (\, 8 + \Omega_j \,)} \right] \left[\frac{2 \, \Omega_j \, (\, 2 + \Omega_j \,) + \Omega_j + (\Omega_j)^2 - 8}{2 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\beta_j^P \, \eta_j \, \Delta_{kj} \, [\, 2 + \Omega_j \,]}{2 \, [\, 8 + \Omega_j \,]} \left[\frac{3 \, \Omega_j + (\Omega_j)^2 - 4}{8 + \Omega_j} \right]$$

$$= \beta_j^S \, \beta_j^P \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2\beta_j^S \, (\, 8 + \Omega_j \,)} \right] \left[\frac{5 \, \Omega_j + 3 \, (\Omega_j)^2 - 8}{2 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\beta_j^P \, \eta_j \, \Delta_{kj} \, [\, 2 + \Omega_j \,]}{2 \, [\, 8 + \Omega_j \,]} \left[\frac{3 \, \Omega_j + (\Omega_j)^2 - 4}{8 + \Omega_j} \right] \; < \; 0. \tag{120}$$

(119) holds because $\Omega_j \in (0, 1)$. (120) holds because

$$5 \, \Omega_{kj} + 3 \, (\Omega_{kj})^2 - 8 \; < \; 0 \;\; \text{and} \;\; 3 \, \Omega_{kj} + (\Omega_{kj})^2 - 4 \; < \; 0. \tag{121}$$

(121) holds because both $5 \, \Omega_{kj} + 3 \, (\Omega_{kj})^2$ and $3 \, \Omega_{kj} + (\Omega_{kj})^2$ increase in $\Omega_{kj} \in (0, 1)$, and thus, $5 \, \Omega_{kj} + 3 \, (\Omega_{kj})^2 < 5 \times 1 + 3 \, (1)^2 = 8$ and $3 \, \Omega_{kj} + (\Omega_{kj})^2 < 3 \times 1 + 1^2 = 4$.

(5) implies that if Sj sells on Pk ($j, k \in \{1, 2\}$) and faces no competition, then:

$$q_{kj}^S \;=\; A_j - b_j^S \, p_{kj}^S \quad \Leftrightarrow \quad p_{kj}^S = \frac{A_j}{b_j^S} - \frac{q_{kj}^S}{b_j^S}. \tag{122}$$

(122) implies that if S1 sells on Pk, S2 sells on Pi ($k, i \in \{1, 2\}$), and each seller faces no competition, then consumer surplus is:

$$CS \;=\; U \left(Q_{k1}^{*S}, Q_{i2}^{*S} \right) - p_{k1}^{*S} \, Q_{k1}^{*S} - p_{i2}^{*S} \, Q_{i2}^{*S} \tag{123}$$

$$= \frac{A_1}{b_1^S} \, Q_{k1}^{*S} + \frac{A_2}{b_2^S} \, Q_{i2}^{*S} - p_{k1}^{*S} \, Q_{k1}^{*S} - p_{i2}^{*S} \, Q_{i2}^{*S} - \frac{1}{2} \left[\frac{\left(Q_{k1}^{*S} \right)^2}{b_1^S \, \Theta_k} + \frac{\left(Q_{i2}^{*S} \right)^2}{b_2^S \, \Theta_i} \right] \tag{124}$$

$$= \left[\frac{A_1}{b_1^S} - p_{k1}^{*S} \right] Q_{k1}^{*S} + \left[\frac{A_2}{b_2^S} - p_{i2}^{*S} \right] Q_{i2}^{*S} - \frac{1}{2} \left[\frac{\left(Q_{k1}^{*S} \right)^2}{b_1^S \, \Theta_k} + \frac{\left(Q_{i2}^{*S} \right)^2}{b_2^S \, \Theta_i} \right]$$

$$= \left[\frac{A_1}{b_1^S} - \frac{A_1}{b_1^S} + \frac{q_{k1}^{*S}}{b_1^S} \right] Q_{k1}^{*S} + \left[\frac{A_2}{b_2^S} - \frac{A_2}{b_2^S} + \frac{q_{i2}^{*S}}{b_2^S} \right] Q_{i2}^{*S} - \frac{1}{2} \left[\frac{\left(Q_{k1}^{*S} \right)^2}{b_1^S \, \Theta_k} + \frac{\left(Q_{i2}^{*S} \right)^2}{b_2^S \, \Theta_i} \right] \tag{125}$$

$$= \left[\frac{A_1}{b_1^S} - \frac{A_1}{b_1^S} + \frac{q_{k1}^{*S}}{b_1^S} \right] \Theta_k \, q_{k1}^{*S} + \left[\frac{A_2}{b_2^S} - \frac{A_2}{b_2^S} + \frac{q_{i2}^{*S}}{b_2^S} \right] \Theta_i \, q_{i2}^{*S} - \frac{1}{2} \left[\frac{\Theta_k \left(q_{k1}^{*S} \right)^2}{b_1^S} + \frac{\Theta_i \left(q_{i2}^{*S} \right)^2}{b_2^S} \right] \tag{126}$$

$$= \frac{\Theta_k}{b_1^S}\left[q_{k1}^{*S}\right]^2+\frac{\Theta_i}{b_2^S}\left[q_{i2}^{*S}\right]^2-\frac{\Theta_k}{2\,b_1^S}\left[q_{k1}^{*S}\right]^2-\frac{\Theta_i}{2\,b_2^S}\left[q_{i2}^{*S}\right]^2 \;=\; \frac{\Theta_k}{2\,b_1^S}\left[q_{k1}^{*S}\right]^2+\frac{\Theta_i}{2\,b_2^S}\left[q_{i2}^{*S}\right]^2 . \tag{127}$$

(125) reflects (122). (126) reflects (11) and $\Theta_k \equiv B_k\,[\,1-f_i\,]$.

Lemma 3 implies that if Sj sells on Pk ($k,i \in \{1,2\}$) and faces no competition, then:

$$q_{kj}^{*S} \;=\; \frac{Q_j^{*S}}{\Theta_k} \;=\; \frac{\widetilde{\Delta}_{kj}}{4}. \tag{128}$$

(127) and (128) imply that if S1 sells on Pk, S2 sells on Pi ($k,i \in \{1,2\}$), and each seller faces no competition, then consumer surplus is:

$$CS \;=\; \frac{\Theta_k}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_{k1}}{4}\right]^2+\frac{\Theta_i}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_{i2}}{4}\right]^2 \;=\; \frac{\Theta_k}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{k1}}{4}\right]^2+\frac{\Theta_i}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{i2}}{4}\right]^2 . \tag{129}$$

The second equality in (129) reflects (10).

(106), (107), and (122) imply that if S1 competes against Pk and S2 sells on Pi and faces no competition ($k,i \in \{1,2\}$), then consumer surplus is:

$$CS \;=\; U\left(Q_{k1}^{*P},Q_{k1}^{*S},Q_{i2}^{*S}\right)-p_{k1}^{*P}\,Q_{k1}^{*P}-p_{k1}^{*S}\,Q_{k1}^{*S}-p_{i2}^{*S}\,Q_{i2}^{*S} \tag{130}$$

$$= \frac{\frac{1}{2}\,\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k}+\frac{1}{2}\,\beta_1^S\,\frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k}-\alpha_1\left[\,\eta_1+\beta_1^S\,\theta_1\,\right]\,Q_{k1}^{*P}}{\left[\eta_1\right]^2-\beta_1^S\,\beta_1^P}$$

$$+\;\frac{\frac{1}{2}\,\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k}+\frac{1}{2}\,\beta_1^P\,\frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k}-\alpha_1\left[\,\theta_1\,\eta_1+\beta_1^P\,\right]\,Q_{k1}^{*S}}{\left[\eta_1\right]^2-\beta_1^S\,\beta_1^P}$$

$$+\;\frac{A_2}{b_2^S}\,Q_{i2}^{*S}-\frac{\left[Q_{i2}^{*S}\right]^2}{2\,b_2^S\,\Theta_i}-p_{k1}^{*P}\,Q_{k1}^{*P}-p_{k1}^{*S}\,Q_{k1}^{*S}-p_{i2}^{*S}\,Q_{i2}^{*S} \tag{131}$$

$$= \frac{\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k}+\frac{1}{2}\,\beta_1^S\,\frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k}+\frac{1}{2}\,\beta_1^P\,\frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k}}{\left[\eta_1\right]^2-\beta_1^S\,\beta_1^P}+\left[\frac{A_2}{b_2^S}-p_{i2}^{*S}\right]Q_{i2}^{*S}-\frac{\left[Q_{i2}^{*S}\right]^2}{2\,b_2^S\,\Theta_i}$$

$$-\;\left[\frac{\alpha_1\left(\,\eta_1+\beta_1^S\,\theta_1\,\right)}{\left(\eta_1\right)^2-\beta_1^S\,\beta_1^P}+p_{k1}^{*P}\right]Q_{k1}^{*P}-\left[\frac{\alpha_1\left(\,\theta_1\,\eta_1+\beta_1^P\,\right)}{\left(\eta_1\right)^2-\beta_1^S\,\beta_1^P}+p_{k1}^{*S}\right]Q_{k1}^{*S}$$

$$= \frac{\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k}+\frac{1}{2}\,\beta_1^S\,\frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k}+\frac{1}{2}\,\beta_1^P\,\frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k}}{\left[\eta_1\right]^2-\beta_1^S\,\beta_1^P}-\left[\frac{\eta_1\,q_{k1}^{*S}+\beta_1^S\,q_{k1}^{*P}}{\left(\eta_1\right)^2-\beta_1^S\,\beta_1^P}\right]Q_{k1}^{*P}-\left[\frac{\eta_1\,q_{k1}^{*P}+\beta_1^P\,q_{k1}^{*S}}{\left(\eta_1\right)^2-\beta_1^S\,\beta_1^P}\right]Q_{k1}^{*S} \tag{132}$$

$$+\;\left[\frac{A_2}{b_2^S}-\frac{A_2}{b_2^S}+\frac{q_{i2}^{*S}}{b_2^S}\right]Q_{i2}^{*S}-\frac{\left[Q_{i2}^{*S}\right]^2}{2\,b_2^S\,\Theta_i} \tag{133}$$

$$= \frac{\eta_1 \, \Theta_k \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{\beta_1^S \, \Theta_k}{2} \left[q_{k1}^{*P} \right]^2 + \frac{\beta_1^P \, \Theta_k}{2} \left[q_{k1}^{*S} \right]^2}{[\eta_1]^2 - \beta_1^S \, \beta_1^P} - \left[\frac{\eta_1 \, q_{k1}^{*S} + \beta_1^S \, q_{k1}^{*P}}{(\eta_1)^2 - \beta_1^S \, \beta_1^P} \right] \Theta_k \, q_{k1}^{*P}$$

$$- \left[\frac{\eta_1 \, q_{k1}^{*P} + \beta_1^P \, q_{k1}^{*S}}{(\eta_1)^2 - \beta_1^S \, \beta_1^P} \right] \Theta_k \, q_{k1}^{*S} + \left[\frac{A_2}{b_2^S} - \frac{A_2}{b_2^S} + \frac{q_{i2}^{*S}}{b_2^S} \right] \Theta_i \, q_{i2}^{*S} - \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S} \tag{134}$$

$$= \Theta_k \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{\beta_1^S}{2} \left[q_{k1}^{*P} \right]^2 + \frac{\beta_1^P}{2} \left[q_{k1}^{*S} \right]^2}{[\eta_1]^2 - \beta_1^S \, \beta_1^P} - \Theta_k \frac{\eta_1 \, q_{k1}^{*S} \, q_{k1}^{*P} + \beta_1^S \left[q_{k1}^{*P} \right]^2}{[\eta_1]^2 - \beta_1^S \, \beta_1^P} - \Theta_k \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \beta_1^P \left[q_{k1}^{*S} \right]^2}{[\eta_1]^2 - \beta_1^S \, \beta_1^P}$$

$$+ \frac{\Theta_i}{b_2^S} \left[q_{i2}^{*S} \right]^2 - \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S}$$

$$= \Theta_k \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{1}{2} \beta_1^S \left[q_{k1}^{*P} \right]^2 + \frac{1}{2} \beta_1^P \left[q_{k1}^{*S} \right]^2 - \eta_1 \, q_{k1}^{*S} \, q_{k1}^{*P} - \beta_1^S \left[q_{k1}^{*P} \right]^2 - \eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} - \beta_1^P \left[q_{k1}^{*S} \right]^2}{[\eta_1]^2 - \beta_1^S \, \beta_1^P}$$

$$+ \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S}$$

$$= \Theta_k \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{1}{2} \beta_1^S \left[q_{k1}^{*P} \right]^2 + \frac{1}{2} \beta_1^P \left[q_{k1}^{*S} \right]^2}{\beta_1^S \, \beta_1^P - [\eta_1]^2} + \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S}. \tag{135}$$

(132) reflects (106) and (107). (133) reflects (122). (134) reflects (11) and $\Theta_k \equiv B_k [1 - f_i]$.

(113), (115), (128), and (135) imply that if S1 competes against Pk and S2 sells on Pi and faces no competition ($k, i \in \{1, 2\}$), then consumer surplus is:

$$CS \;=\; \Theta_k \, \varsigma_{k1} + \frac{\Theta_i}{2 \, b_2^S} \left[\frac{\widetilde{\Delta}_{i2}}{4} \right]^2 \;=\; \Theta_k \, \varsigma_{k1} + \frac{\Theta_i}{2 \, \beta_2^S \, [\, 1 - \Omega_2 \,]} \left[\frac{\widetilde{\Delta}_{i2}}{4} \right]^2 . \tag{136}$$

The second equality in (136) reflects (10).

(12) and (115) imply that:

$$\varsigma_{kj} \;>\; \frac{1}{2 \, \beta_j^S \, [\, 1 - \Omega_j \,]} \left[\frac{\widetilde{\Delta}_{kj}}{4} \right]^2$$

$$\Leftrightarrow \;\; \frac{1}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \left\{ \frac{\eta_j \, [\, 2 + \Omega_j \,] \; \Delta_{kj}}{8 + \Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \right.$$

$$\left. + \frac{\beta_j^S}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \; \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 + \frac{\beta_j^P}{2} \left[\frac{(\, 2 + \Omega_j \,) \; \Delta_{kj}}{8 + \Omega_j} \right]^2 \right\} \tag{137}$$

$$> \;\; \frac{1}{32 \, \beta_j^S \, [\, 1 - \Omega_j \,]} \left[\Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj} \right]^2$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]+\frac{\beta_j^S}{2}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]^2$$

$$+\;\frac{\beta_j^P}{2}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]^2\;>\;\frac{\beta_j^P}{32}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{2\,[\,8+\Omega_j\,]}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,[\,8+\Omega_j\,]^2}$$

$$+\;\frac{\beta_j^S}{8}\left[\overline{\Delta}_{kj}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right]^2\;>\;\frac{\beta_j^P}{32}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}$$

$$+\;\frac{\beta_j^S}{4}\left[\left(\overline{\Delta}_{kj}\right)^2+\frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2}+\frac{2\,\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right]\;>\;\frac{\beta_j^P}{16}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}+\;\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4}$$

$$+\;\frac{\beta_j^S\,\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}+\frac{2\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}\;>\;\frac{\beta_j^P}{16}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{4\,\beta_j^S\left[\eta_j\right]^2[\,2+\Omega_j\,]^2+4\left[\beta_j^S\right]^2\beta_j^P\,[\,2+\Omega_j\,]^2+\beta_j^S\left[\eta_j\right]^2[\,2+\Omega_j\,]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}\,[\Delta_{kj}]^2$$

$$+\;\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4}\;>\;\frac{\beta_j^P}{16}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{4\left[\eta_j\right]^2+4\,\beta_j^S\,\beta_j^P+\left[\eta_j\right]^2}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2$$

$$+\;\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4}\;>\;\frac{\beta_j^P}{16}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \quad \frac{\left[5\left(\eta_j\right)^2+4\,\beta_j^S\,\beta_j^P\right][\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}+\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2$$

$$>\;\frac{\beta_j^P}{4}\left[[\Delta_{kj}]^2+\frac{\left(\eta_j\right)^2\left(\overline{\Delta}_{kj}\right)^2}{\left(\beta_j^P\right)^2}+\frac{2\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^P}\right]$$

$$\Leftrightarrow \quad \frac{\beta_j^S \, \beta_j^P \, [\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 8 + \Omega_j\,]^2} + \frac{6\,\eta_j \, [\, 2 + \Omega_j\,] \; \Delta_{kj} \, \overline{\Delta}_{kj}}{8 + \Omega_j} + \beta_j^S \, \big[\overline{\Delta}_{kj}\big]^2$$

$$> \; \frac{\beta_j^P \, [\Delta_{kj}]^2}{4} + \frac{\big[\eta_j\big]^2 \, \big[\overline{\Delta}_{kj}\big]^2}{4\,\beta_j^P} + \frac{\eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{2}$$

$$\Leftrightarrow \quad \frac{\beta_j^P \, [\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 \, [\Delta_{kj}]^2}{[\, 8 + \Omega_j\,]^2} - \frac{\beta_j^P \, [\Delta_{kj}]^2}{4} + \frac{6\,\eta_j \, [\, 2 + \Omega_j\,] \; \Delta_{kj} \, \overline{\Delta}_{kj}}{8 + \Omega_j} - \frac{\eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{2}$$

$$+ \; \beta_j^S \, \big[\overline{\Delta}_{kj}\big]^2 - \frac{\big[\eta_j\big]^2 \, \big[\overline{\Delta}_{kj}\big]^2}{4\,\beta_j^P} \; > \; 0$$

$$\Leftrightarrow \quad \frac{4\,[\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 - [\, 8 + \Omega_j\,]^2}{4\,[\, 8 + \Omega_j\,]^2} \beta_j^P \, [\Delta_{kj}]^2 + \frac{12\,[\, 2 + \Omega_j\,] - [\, 8 + \Omega_j\,]}{2\,[\, 8 + \Omega_j\,]} \eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}$$

$$+ \; \frac{4\,\beta_j^P \, \beta_j^S - \big[\eta_j\big]^2}{4\,\beta_j^P} \, \big[\overline{\Delta}_{kj}\big]^2 \; > \; 0$$

$$\Leftrightarrow \quad \frac{4\,[\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 - [\, 8 + \Omega_j\,]^2}{4\,[\, 8 + \Omega_j\,]^2} \beta_j^P \, [\Delta_{kj}]^2 + \frac{[\, 16 + 11\,\Omega_j\,] \, \eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{2\,[\, 8 + \Omega_j\,]}$$

$$+ \; \frac{\beta_j^P \, \beta_j^S \, [\, 4 - \Omega_j\,] \, \big[\overline{\Delta}_{kj}\big]^2}{4\,\beta_j^P} \; > \; 0. \tag{138}$$

(138) holds because $\Omega_j \in (0, 1)$ and

$$4\,[\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 - [\, 8 + \Omega_j\,]^2 \; > \; 0 \quad \Leftrightarrow \quad 4\,[\, 5\,\Omega_j + 4\,] \, [\, 2 + \Omega_j\,]^2 \; > \; [\, 8 + \Omega_j\,]^2$$

$$\Leftrightarrow \quad 4\,[\, 5\,\Omega_j + 4\,] \, \big[\, 4 + (\Omega_j)^2 + 4\,\Omega_j\,\big] \; > \; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad [\, 5\,\Omega_j + 4\,] \, \big[\, 16 + 4\,(\Omega_j)^2 + 16\,\Omega_j\,\big] \; > \; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad 5\,\Omega_j \, \big[\, 16 + 4\,(\Omega_j)^2 + 16\,\Omega_j\,\big] + 4\,\big[\, 16 + 4\,(\Omega_j)^2 + 16\,\Omega_j\,\big] \; > \; 64 + [\Omega_j]^2 + 16\,\Omega_j$$

$$\Leftrightarrow \quad 80\,\Omega_j + 20\,[\Omega_j]^3 + 80\,[\Omega_j]^2 + 64 + 16\,[\Omega_j]^2 + 64\,\Omega_j - 64 - [\Omega_j]^2 - 16\,\Omega_j \; > \; 0$$

$$\Leftrightarrow \quad 128\,\Omega_j + 20\,[\Omega_j]^3 + 95\,[\Omega_j]^2 \; > \; 0. \tag{139}$$

(139) holds because $\Omega_j \in (0, 1)$.

Proposition 1 implies that each seller competes against P under MP. Therefore, (114) implies that:

$$CS^M \; = \; \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}, \tag{140}$$

where ς_{Pj} is given by (115).

First suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$)

under PC, where $\widetilde{\Theta}$ denotes $\widetilde{\text{P}}$'s platform strength. (120) and $\frac{\widetilde{c}_j^P}{c_j^P} > 1$ imply that:

$$\frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}} \ > \ 1. \tag{141}$$

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} > \max\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}$.

Because $\frac{\widetilde{\Theta}}{\Theta} > \phi_{\widetilde{P}2}$, Proposition 2 implies that each seller competes against $\widetilde{\text{P}}$ under PC. Therefore, (114) implies that:

$$CS \ = \ \widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \widetilde{\Theta}\,\varsigma_{\widetilde{P}2}, \tag{142}$$

where $\varsigma_{\widetilde{P}j}$ is given by (115).

Because $\frac{\widetilde{\Theta}}{\Theta} > \frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}j} > \Theta\,\varsigma_{Pj}$. Therefore, (140) and (142) imply that $CS > CS^M$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}\right)$.

First suppose $\phi_{\widetilde{P}1} < \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}\right\}$.

If $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\}\right)$, then $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \phi_{\widetilde{P}2}\right)$. Proposition 2 implies that S1 competes against $\widetilde{\text{P}}$ whereas S2 sells on P and faces no competition under PC. Therefore, (136) implies that:

$$CS \ = \ \widetilde{\Theta}\,\varsigma_{\widetilde{P}1} + \frac{\Theta}{2\,\beta_2^S\,[\,1 - \Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \tag{143}$$

Because $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}1} < \Theta\,\varsigma_{P1}$. (137) implies that $\frac{1}{2\,\beta_2^S[1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2 < \varsigma_{P2}$. Therefore, (140) and (143) imply that $CS < CS^M$ in this case.

If $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$, Proposition 2 implies that each seller sells on $\widetilde{\text{P}}$ and faces no competition under PC. Therefore, (129) implies that:

$$CS \ = \ \frac{\widetilde{\Theta}}{2\,\beta_1^S\,[\,1 - \Omega_1\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}1}}{4}\right]^2 + \frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1 - \Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2. \tag{144}$$

(137) implies that $\frac{\widetilde{\Theta}}{2\,\beta_j^S[1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \widetilde{\Theta}\,\varsigma_{\widetilde{P}j}$. Because $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}}$, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}j} < \Theta\,\varsigma_{Pj}$. Therefore, $\frac{\widetilde{\Theta}}{2\,\beta_j^S[1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \Theta\,\varsigma_{Pj}$. Therefore, (140) and (144) imply that $CS < CS^M$ in this case.

Next suppose $\phi_{\widetilde{P}1} \geq \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}\right\}$. (91) implies that $\phi_{\widetilde{P}2} > \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}\right\}$. Therefore, $\min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\} \ = \ \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}\right\}$. Because $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \min\left\{\frac{\varsigma_{P1}}{\varsigma_{\widetilde{P}1}}, \frac{\varsigma_{P2}}{\varsigma_{\widetilde{P}2}}\right\}\right)$, $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$.

Proposition 2 implies that each seller sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, consumer surplus is given by (144). Therefore, (140) and (144) imply that $CS < CS^M$ in this case.

Next suppose P faces a symmetric competing platform $\widetilde{\mathrm{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 2 implies that each seller is indifferent between selling on P and selling on $\widetilde{\mathrm{P}}$ and each seller faces no competition under PC. Therefore, (129) implies that:

$$CS = \frac{1}{2}\,\frac{\widetilde{\Theta}}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}1}}{4}\right]^2 + \frac{1}{2}\,\frac{\Theta}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{P1}}{4}\right]^2$$
$$+ \frac{1}{2}\,\frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2 + \frac{1}{2}\,\frac{\Theta}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \tag{145}$$

(137) implies that $\frac{\Theta}{2\,\beta_j^S[1-\Omega_j]}\left[\frac{\widetilde{\Delta}_{Pj}}{4}\right]^2 < \Theta\,\varsigma_{Pj}$ and $\frac{\widetilde{\Theta}}{2\,\beta_j^S[1-\Omega_j]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \widetilde{\Theta}\,\varsigma_{\widetilde{P}j}$. Because $\frac{\widetilde{\Theta}}{\Theta} = 1$, (141) implies that $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}}$, and thus, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}j} < \Theta\,\varsigma_{Pj}$. Therefore, $\frac{\widetilde{\Theta}}{2\,\beta_j^S[1-\Omega_j]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 < \Theta\,\varsigma_{Pj}$. Therefore, for $j \in \{1,2\}$

$$\frac{1}{2}\,\frac{\widetilde{\Theta}}{2\,\beta_j^S\,[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}j}}{4}\right]^2 + \frac{1}{2}\,\frac{\Theta}{2\,\beta_j^S\,[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{Pj}}{4}\right]^2 < \frac{\Theta}{2}\,\varsigma_{Pj} + \frac{\Theta}{2}\,\varsigma_{Pj} = \Theta\,\varsigma_{Pj}. \tag{146}$$

Therefore, (140), (145), and (146) imply that $CS < CS^M$ in this case.

Finally, suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P1}}, 1\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,1, \phi_1\,)$. Proposition 2 implies that each seller sells on P and faces no competition under PC. Therefore, (129) implies that:

$$CS = \frac{\Theta}{2\,\beta_1^S\,[\,1-\Omega_1\,]}\left[\frac{\widetilde{\Delta}_{P1}}{4}\right]^2 + \frac{\Theta}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{P2}}{4}\right]^2. \tag{147}$$

(137) implies that $\frac{1}{2\,\beta_j^S[1-\Omega_j]}\left[\frac{\widetilde{\Delta}_{Pj}}{4}\right]^2 < \varsigma_{Pj}$ for $j \in \{1,2\}$. Therefore, (140) and (147) imply that $CS < CS^M$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P2}}, \frac{1}{\phi_{P1}}\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,\phi_1, \phi_2\,)$. Proposition 2 implies that S1 competes against P whereas S2 sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, (136) implies that:

$$CS \;=\; \Theta\,\varsigma_{P1} + \frac{\widetilde{\Theta}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2. \tag{148}$$

(137) implies that $\frac{\widetilde{\Theta}}{2\,\beta_2^S[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2 < \widetilde{\Theta}\,\varsigma_{\widetilde{P}2}$. (141) and $\frac{\widetilde{\Theta}}{\Theta} < 1$ imply that $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{Pj}}{\varsigma_{\widetilde{P}j}}$, and thus, $\widetilde{\Theta}\,\varsigma_{\widetilde{P}j} < \Theta\,\varsigma_{Pj}$. Therefore, $\frac{\widetilde{\Theta}}{2\,\beta_2^S[\,1-\Omega_2\,]}\left[\frac{\widetilde{\Delta}_{\widetilde{P}2}}{4}\right]^2 < \Theta\,\varsigma_{P2}$. Therefore, (140) and (148) imply that $CS < CS^M$ in this case. ■

Proposition 6. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$). Then $p_1^S \geq p_1^{SM}$ and $p_2^S > p_2^{SM}$, where the first inequality holds strictly unless $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1}\right)$.*

Proof. (107) and (113) imply that if Sj sells on Pk and competes against Pk ($j, k \in \{1,2\}$), then in equilibrium:

$$\begin{aligned}
p_{kj}^{*S} \;&=\; \frac{1}{[\eta_j]^2 - \beta_j^S\,\beta_j^P}\left\{\eta_j\left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\right] + \frac{\beta_j^P\,[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j} - \alpha_j\left[\,\theta_j\,\eta_j + \beta_j^P\,\right]\right\} \\
&=\; \frac{1}{[\eta_j]^2 - \beta_j^S\,\beta_j^P}\left[\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{(\eta_j)^2\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)} + \frac{\beta_j^P\,(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j} - \alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right)\right] \\
&=\; \frac{1}{[\eta_j]^2 - \beta_j^S\,\beta_j^P}\left[\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{(\eta_j)^2 + 2\,\beta_j^S\,\beta_j^P}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\,(\,2+\Omega_j\,)\;\Delta_{kj} - \alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right)\right] \\
&=\; \frac{1}{[\eta_j]^2 - \beta_j^S\,\beta_j^P}\left[\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{\beta_j^S\,\beta_j^P\,(\,2+\Omega_j\,)}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\,(\,2+\Omega_j\,)\;\Delta_{kj} - \alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right)\right] \\
&=\; \frac{1}{[\eta_j]^2 - \beta_j^S\,\beta_j^P}\left[\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\;\Delta_{kj}}{2\,(\,8+\Omega_j\,)} - \alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right)\right] \\
&=\; -\frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\;\Delta_{kj}}{2\,(\,8+\Omega_j\,)} - \alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right)\right] \\
&=\; \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\alpha_j\left(\,\theta_j\,\eta_j + \beta_j^P\,\right) - \frac{\eta_j\,\overline{\Delta}_{kj}}{2} - \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\;\Delta_{kj}}{2\,(\,8+\Omega_j\,)}\right].
\end{aligned} \tag{149}$$

(122) and (128) imply that if Sj sells on Pk ($j, k \in \{1,2\}$) and faces no competition, then

in equilibrium:

$$p_{kj}^{*S} \;=\; \frac{A_j}{b_j^S} - \frac{\widetilde{\Delta}_{kj}}{4\,b_j^S} \;=\; \frac{4\,A_j - \widetilde{\Delta}_{kj}}{4\,b_j^S} \;=\; \frac{4\,A_j - \widetilde{\Delta}_{kj}}{4\,\beta_j^S\,[\,1-\Omega_j\,]}. \tag{150}$$

The last equality in (150) reflects (10).

Let p_j^{SM} denote Sj's equilibrium price under MP and p_j^S denote Sj's equilibrium price under PC. Proposition 1 implies that Sj ($j \in \{1,2\}$) competes against P under MP. Therefore, (149) implies that:

$$p_j^{SM} \;=\; \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\alpha_j\left(\theta_j\,\eta_j + \beta_j^P\right) - \frac{\eta_j\,\overline{\Delta}_{Pj}}{2} - \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\;\Delta_{Pj}}{2\,(\,8+\Omega_j\,)}\right]. \tag{151}$$

First suppose P faces a competing platform $\widetilde{\text{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.

Proposition 2 implies that Sj ($j \in \{1,2\}$) competes against $\widetilde{\text{P}}$ under PC. Therefore, (149) implies that:

$$p_j^{S} \;=\; \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\alpha_j\left(\theta_j\,\eta_j + \beta_j^P\right) - \frac{\eta_j\,\overline{\Delta}_{\widetilde{P}j}}{2} - \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\;\Delta_{\widetilde{P}j}}{2\,(\,8+\Omega_j\,)}\right]. \tag{152}$$

Observe that:

$$\frac{\partial\left(\frac{\eta_j\,\overline{\Delta}_{kj}}{2} + \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\,\Delta_{kj}}{2(\,8+\Omega_j\,)}\right)}{\partial c_{kj}^P} \;=\; \frac{\eta_j}{2}\,\frac{\partial\overline{\Delta}_{kj}}{\partial c_{kj}^P} + \frac{\beta_j^P\,[\,2+\Omega_j\,]^2}{2\,[\,8+\Omega_j\,]}\,\frac{\partial\Delta_{kj}}{\partial c_{kj}^P} \;=\; -\frac{\eta_j\,\beta_j^P}{2} + \frac{\eta_j\,\beta_j^P\,[\,2+\Omega_j\,]^2}{2\,[\,8+\Omega_j\,]}$$

$$\overset{s}{=}\; -1 + \frac{[\,2+\Omega_j\,]^2}{8+\Omega_j} \;=\; \frac{[\,2+\Omega_j\,]^2 - [\,8+\Omega_j\,]}{8+\Omega_j} \;=\; \frac{4+4\,\Omega_j + [\Omega_j]^2 - 8 - \Omega_j}{8+\Omega_j} \;=\; \frac{3\,\Omega_j + [\Omega_j]^2 - 4}{8+\Omega_j} \;<\; 0. \tag{153}$$

The inequality in (153) holds because $3\,\Omega_j + [\Omega_j]^2 - 4$ increases in $\Omega_j \in (0,1)$, and therefore, $3\,\Omega_j + [\Omega_j]^2 - 4 < \max 3\,\Omega_j + [\Omega_j]^2 - 4 = 3*1 + [1]^2 - 4 = 0$.

(151) - (153) imply that $p_j^S > p_j^{SM}$ in this case because $\frac{\widetilde{c}_j^P}{c_j^P} > 1$.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in (\,\phi_1, \phi_2\,)$.

Proposition 2 implies that S1 competes against $\widetilde{\text{P}}$ whereas S2 sells on P and faces no competition under PC. Therefore, (149) implies that:

$$p_1^{S} \;=\; \frac{1}{\beta_1^S\,\beta_1^P\,[\,1-\Omega_1\,]}\left[\alpha_1\left(\theta_1\,\eta_1 + \beta_1^P\right) - \frac{\eta_1\,\overline{\Delta}_{\widetilde{P}1}}{2} - \frac{\beta_1^P\,(\,2+\Omega_1\,)^2\;\Delta_{\widetilde{P}1}}{2\,(\,8+\Omega_1\,)}\right]. \tag{154}$$

(151), (153), and (154) imply that $p_1^S > p_1^{SM}$ in this case because $\frac{\widetilde{c}_j^P}{c_j^P} > 1$. (150) implies that:

$$p_2^S \;=\; \frac{4\,A_2 - \widetilde{\Delta}_{P2}}{4\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{155}$$

(151) and (155) imply that:

$$p_2^S \;>\; p_2^{SM} \;\;\Leftrightarrow\;\; \frac{4\,A_2 - \widetilde{\Delta}_{P2}}{4\,\beta_2^S\,[\,1-\Omega_2\,]} \;>\; \frac{1}{\beta_2^S\,\beta_2^P\,[\,1-\Omega_2\,]}\left[\alpha_2\left(\theta_2\,\eta_2+\beta_2^P\right) - \frac{\eta_2\,\overline{\Delta}_{P2}}{2} - \frac{\beta_2^P\,(\,2+\Omega_2\,)^2\,\Delta_{P2}}{2\,(\,8+\Omega_2\,)}\right] \tag{156}$$

$$\Leftrightarrow\;\; \beta_2^P\left[\,4\,A_2 - \widetilde{\Delta}_{P2}\,\right] \;>\; 4\left[\alpha_2\left(\theta_2\,\eta_2+\beta_2^P\right) - \frac{\eta_2\,\overline{\Delta}_{P2}}{2} - \frac{\beta_2^P\,(\,2+\Omega_2\,)^2\,\Delta_{P2}}{2\,(\,8+\Omega_2\,)}\right]$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - \beta_2^P\,\widetilde{\Delta}_{P2} \;>\; 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] - 2\,\eta_2\,\overline{\Delta}_{P2} - \frac{2\,\beta_2^P\,[\,2+\Omega_2\,]^2\,\Delta_{P2}}{[\,8+\Omega_2\,]}$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - \beta_2^P\left[\,\Delta_{P2} + \frac{\eta_2}{\beta_2^P}\,\overline{\Delta}_{P2}\,\right] \;>\; 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] - 2\,\eta_2\,\overline{\Delta}_{P2} - \frac{2\,\beta_2^P\,[\,2+\Omega_2\,]^2\,\Delta_{P2}}{[\,8+\Omega_2\,]} \tag{157}$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - \beta_2^P\,\Delta_{P2} - \eta_2\,\overline{\Delta}_{P2} \;>\; 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] - 2\,\eta_2\,\overline{\Delta}_{P2} - \frac{2\,\beta_2^P\,[\,2+\Omega_2\,]^2\,\Delta_{P2}}{[\,8+\Omega_2\,]}$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] + \frac{2\,\beta_2^P\,[\,2+\Omega_2\,]^2\,\Delta_{P2}}{[\,8+\Omega_2\,]} - \beta_2^P\,\Delta_{P2} + 2\,\eta_2\,\overline{\Delta}_{P2} - \eta_2\,\overline{\Delta}_{P2} \;>\; 0$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] + \frac{2\,[\,2+\Omega_2\,]^2 - [\,8+\Omega_2\,]}{[\,8+\Omega_2\,]}\beta_2^P\,\Delta_{P2} + \eta_2\,\overline{\Delta}_{P2} \;>\; 0$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] + \frac{2\left[\,4+4\,\Omega_2+(\Omega_2)^2\,\right] - 8 - \Omega_2}{[\,8+\Omega_2\,]}\beta_2^P\,\Delta_{P2} + \eta_2\,\overline{\Delta}_{P2} \;>\; 0$$

$$\Leftrightarrow\;\; 4\,\beta_2^P\,A_2 - 4\,\alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right] + \frac{7\,\Omega_2 + 2\,[\Omega_2]^2}{[\,8+\Omega_2\,]}\beta_2^P\,\Delta_{P2} + \eta_2\,\overline{\Delta}_{P2} \;>\; 0$$

$$\Leftrightarrow\;\; \frac{7\,\Omega_2 + 2\,[\Omega_2]^2}{[\,8+\Omega_2\,]}\beta_2^P\,\Delta_{P2} + \eta_2\,\overline{\Delta}_{P2} \;>\; 0. \tag{158}$$

(157) reflects (12). (158) reflects $\beta_2^P\,A_2 \,=\, \alpha_2\left[\,\theta_2\,\eta_2+\beta_2^P\,\right]$ from (7). (158) holds because $\Omega_2 \in (0,1)$.

<u>Case III</u>. $\frac{\widetilde{\Theta}}{\Theta} \in (\,1, \phi_1\,)$.

Proposition 2 implies that Sj ($j \in \{1,2\}$) sells on $\widetilde{\text{P}}$ and faces no competition under PC. Therefore, (150) implies that:

$$p_j^S \;=\; \frac{4\,A_j - \widetilde{\Delta}_{\widetilde{P}j}}{4\,\beta_j^S\,[\,1-\Omega_j\,]}. \tag{159}$$

(118) and (156) imply that:

$$\frac{4\,A_j - \widetilde{\Delta}_{\widetilde{P}j}}{4\,\beta_j^S\,[\,1-\Omega_j\,]} \;=\; \frac{4\,A_j - \widetilde{\Delta}_{Pj}}{4\,\beta_j^S\,[\,1-\Omega_j\,]} \;>\; \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\alpha_j\left(\theta_j\,\eta_j+\beta_j^P\right) - \frac{\eta_j\,\overline{\Delta}_{Pj}}{2} - \frac{\beta_j^P\,(\,2+\Omega_j\,)^2\,\Delta_{Pj}}{2\,(\,8+\Omega_j\,)}\right]. \tag{160}$$

(151), (159), and (160) imply that $p_j^S > p_j^{SM}$ in this case.

Next suppose P faces a symmetric competing platform $\widetilde{\mathrm{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta}=1$) under PC. Proposition 2 implies that Sj ($j\in\{1,2\}$) is indifferent between selling on P and selling on $\widetilde{\mathrm{P}}$ and Sj faces no competition under PC. (150) implies that if Sj sells on P, then:

$$p_j^S \;=\; \frac{4\,A_j - \widetilde{\Delta}_{Pj}}{4\,\beta_j^S\,[\,1-\Omega_j\,]}; \tag{161}$$

if Sj sells on $\widetilde{\mathrm{P}}$, then:

$$p_j^S \;=\; \frac{4\,A_j - \widetilde{\Delta}_{\widetilde{P}j}}{4\,\beta_j^S\,[\,1-\Omega_j\,]}. \tag{162}$$

(151), (160), (161), and (162) imply that $p_j^S > p_j^{SM}$ in this case.

Finally, suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,1, \phi_1\,)$. Proposition 2 implies that Sj ($j\in\{1,2\}$) sells on P and faces no competition under PC. Therefore, (150) implies that p_j^S is given by (161). (151), (160), and (161) imply that $p_j^S > p_j^{SM}$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1}\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,\phi_1, \phi_2\,)$. Proposition 2 implies that S1 competes against P whereas S2 sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, (149) implies that:

$$p_1^S \;=\; \frac{1}{\beta_1^S\,\beta_1^P\,[\,1-\Omega_1\,]}\left[\alpha_1\left(\theta_1\,\eta_1+\beta_1^P\right) - \frac{\eta_1\,\overline{\Delta}_{P1}}{2} - \frac{\beta_1^P\,(\,2+\Omega_1\,)^2\,\Delta_{P1}}{2\,(\,8+\Omega_1\,)}\right]. \tag{163}$$

(151) and (163) imply that $p_1^S = p_1^{SM}$ in this case. (150) implies that:

$$p_2^S \;=\; \frac{4\,A_2 - \widetilde{\Delta}_{\widetilde{P}2}}{4\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{164}$$

(151), (160), and (164) imply that $p_2^S > p_2^{SM}$ in this case. ■

Proposition 7. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform* $\widetilde{\text{P}}$ *that is a weaker seller than P (i.e.,* $\frac{\widetilde{c}_j^P}{c_j^P} > 1$*). Then* $w_j < w_j^M$ *if* $\frac{\widetilde{\Theta}}{\Theta} > \phi_j$ *whereas* $w_j \geq w_j^M$ *if* $\frac{\widetilde{\Theta}}{\Theta} < \phi_j$ *(*$j \in \{1, 2\}$*), where the equality holds when* $j = 1$ *and* $\frac{\widetilde{\Theta}}{\Theta} < \frac{1}{\phi_1}$*.*

Proof**.** Let w_j denote the commission Sj faces under PC and w_j^M denote the commission Sj faces under MP ($j \in \{1, 2\}$). Proposition 1 implies that Sj ($j \in \{1, 2\}$) competes against P under MP. Lemma 5 implies that:

$$w_j^M \;=\; \frac{1}{2\,[\,1 - \Omega_j\,]} \left\{ \frac{\Omega_j\,\overline{\Delta}_{Pj}}{\eta_j} + \frac{\left[\,8 + (\Omega_j)^2\,\right]\,\Delta_{Pj}}{\beta_j^S\,[\,8 + \Omega_j\,]} \right\}. \tag{165}$$

First suppose P faces a competing platform $\widetilde{\text{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.

Proposition 2 implies that Sj ($j \in \{1, 2\}$) competes against $\widetilde{\text{P}}$ under PC. Lemma 5 implies that:

$$w_j \;=\; \frac{1}{2\,[\,1 - \Omega_j\,]} \left\{ \frac{\Omega_j\,\overline{\Delta}_{\widetilde{P}j}}{\eta_j} + \frac{\left[\,8 + (\Omega_j)^2\,\right]\,\Delta_{\widetilde{P}j}}{\beta_j^S\,[\,8 + \Omega_j\,]} \right\}. \tag{166}$$

Observe that:

$$\frac{\partial\left(\frac{\Omega_j\,\overline{\Delta}_{kj}}{\eta_j} + \frac{\left[\,8+(\Omega_j)^2\,\right]\Delta_{kj}}{\beta_j^S[\,8+\Omega_j\,]} \right)}{\partial c_{kj}^P} \;=\; \frac{\Omega_j}{\eta_j}\,\frac{\partial\overline{\Delta}_{kj}}{\partial c_{kj}^P} + \frac{\left[\,8 + (\Omega_j)^2\,\right]}{\beta_j^S\,[\,8 + \Omega_j\,]}\,\frac{\partial\Delta_{kj}}{\partial c_{kj}^P} \;=\; -\,\frac{\Omega_j\,\beta_j^P}{\eta_j} + \frac{\eta_j\left[\,8 + (\Omega_j)^2\,\right]}{\beta_j^S\,[\,8 + \Omega_j\,]}$$

$$= \;\frac{\left[\eta_j\right]^2\left[\,8 + (\Omega_j)^2\,\right] - \Omega_j\,\beta_j^P\,\beta_j^S\,[\,8 + \Omega_j\,]}{\eta_j\,\beta_j^S\,[\,8 + \Omega_j\,]} \;=\; \frac{\left[\eta_j\right]^2\left[\,8 + (\Omega_j)^2\,\right] - \left[\eta_j\right]^2[\,8 + \Omega_j\,]}{\eta_j\,\beta_j^S\,[\,8 + \Omega_j\,]}$$

$$\stackrel{s}{=}\; \frac{8 + [\Omega_j]^2 - 8 - \Omega_j}{\eta_j\,\beta_j^S\,[\,8 + \Omega_j\,]} \;=\; -\frac{\Omega_j\,[\,1 - \Omega_j\,]}{\eta_j\,\beta_j^S\,[\,8 + \Omega_j\,]} \;<\; 0. \tag{167}$$

The inequality in (167) holds because $\Omega_j \in (0, 1)$. (165) - (167) imply that $w_j < w_j^M$ if $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$ because $\frac{\widetilde{c}_j^P}{c_j^P} > 1$.

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in (\,\phi_1, \phi_2\,)$.

Proposition 2 implies that S1 competes against $\widetilde{\text{P}}$ whereas S2 sells on P and faces no competition under PC. Lemmas 2 and 5 imply that:

$$w_1 \;=\; \frac{1}{2\,[\,1-\Omega_1\,]}\left\{\frac{\Omega_1\,\overline{\Delta}_{\widetilde{P}1}}{\eta_1}+\frac{\left[\,8+(\Omega_1)^2\,\right]\,\Delta_{\widetilde{P}1}}{\beta_1^S\,[\,8+\Omega_1\,]}\right\} \;\text{ and }\; w_2 \;=\; \frac{\widetilde{\Delta}_{P2}}{2\,b_2^S} \;=\; \frac{\widetilde{\Delta}_{P2}}{2\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{168}$$

The last equality in (168) reflects (10). (165), (167), and (168) imply that $w_1 < w_1^M$ if $\frac{\widetilde{\Theta}}{\Theta} \in (\,\phi_1, \phi_2\,)$ because $\frac{\widetilde{c}_1^P}{c_1^P} > 1$. (165) and (168) imply that:

$$w_2 \;>\; w_2^M \;\;\Leftrightarrow\;\; \frac{\widetilde{\Delta}_{P2}}{2\,\beta_2^S\,[\,1-\Omega_2\,]} \;>\; \frac{1}{2\,[\,1-\Omega_2\,]}\left\{\frac{\Omega_2\,\overline{\Delta}_{P2}}{\eta_2}+\frac{\left[\,8+(\Omega_2)^2\,\right]\,\Delta_{P2}}{\beta_2^S\,[\,8+\Omega_2\,]}\right\} \tag{169}$$

$$\Leftrightarrow\;\; \widetilde{\Delta}_{P2} \;>\; \frac{\beta_2^S\,\Omega_2\,\overline{\Delta}_{P2}}{\eta_2}+\frac{\left[\,8+(\Omega_2)^2\,\right]\,\Delta_{P2}}{8+\Omega_2}$$

$$\Leftrightarrow\;\; \Delta_{P2}+\frac{\eta_2}{\beta_2^P}\,\overline{\Delta}_{P2} \;>\; \frac{\beta_2^S\,\Omega_2\,\overline{\Delta}_{P2}}{\eta_2}+\frac{\left[\,8+(\Omega_2)^2\,\right]\,\Delta_{P2}}{8+\Omega_2} \tag{170}$$

$$\Leftrightarrow\;\; \frac{8+\Omega_2-\left[\,8+(\Omega_2)^2\,\right]}{8+\Omega_2}\,\Delta_{P2}+\frac{[\eta_2]^2-\beta_2^P\,\beta_2^S\,\Omega_2}{\beta_2^P\,\eta_2}\,\overline{\Delta}_{P2} \;>\; 0$$

$$\Leftrightarrow\;\; \frac{\Omega_2-[\Omega_2]^2}{8+\Omega_2}\,\Delta_{P2}+\frac{[\eta_2]^2-[\eta_2]^2}{\beta_2^P\,\eta_2}\,\overline{\Delta}_{P2} \;>\; 0 \;\;\Leftrightarrow\;\; \frac{\Omega_2\,[\,1-\Omega_2\,]}{8+\Omega_2}\,\Delta_{P2} \;>\; 0. \tag{171}$$

(170) reflects (12). The last inequality in (171) reflects $\Omega_2 \in (0,1)$.

<u>Case III</u>. $\frac{\widetilde{\Theta}}{\Theta} \in (\,1, \phi_1\,)$.

Proposition 2 implies that Sj ($j \in \{1,2\}$) sells on $\widetilde{\text{P}}$ and faces no competition under PC. Lemma 2 and (10) imply that:

$$w_j \;=\; \frac{\widetilde{\Delta}_{\widetilde{P}j}}{2\,b_j^S} \;=\; \frac{\widetilde{\Delta}_{\widetilde{P}j}}{2\,\beta_j^S\,[\,1-\Omega_j\,]}. \tag{172}$$

(118) and (169) imply that

$$\frac{\widetilde{\Delta}_{\widetilde{P}j}}{2\,\beta_j^S\,[\,1-\Omega_j\,]} \;=\; \frac{\widetilde{\Delta}_{Pj}}{2\,\beta_j^S\,[\,1-\Omega_j\,]} \;>\; \frac{1}{2\,[\,1-\Omega_j\,]}\left\{\frac{\Omega_j\,\overline{\Delta}_{Pj}}{\eta_j}+\frac{\left[\,8+(\Omega_j)^2\,\right]\,\Delta_{Pj}}{\beta_j^S\,[\,8+\Omega_j\,]}\right\}. \tag{173}$$

Therefore, (165), (172), and (173) imply that $w_j > w_j^M$ if $\frac{\widetilde{\Theta}}{\Theta} \in (\,1, \phi_1\,)$.

Next suppose P faces a symmetric competing platform $\widetilde{\text{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 2 implies that Sj ($j \in \{1,2\}$) is indifferent between selling on P and selling on $\widetilde{\text{P}}$ and

Sj faces no competition under PC. If Sj sells on P, Lemma 2 and (10) imply that:

$$w_j \;=\; \frac{\widetilde{\Delta}_{Pj}}{2\, b_j^S} \;=\; \frac{\widetilde{\Delta}_{Pj}}{2\, \beta_j^S \left[\, 1 - \Omega_j \,\right]}. \tag{174}$$

(165), (169), and (174) imply that $w_j > w_j^M$ in this case. If Sj sells on $\widetilde{\text{P}}$, Lemma 2 and (10) imply that w_j is given by (172). (165), (172), and (173) imply that $w_j > w_j^M$ in this case.

Finally, suppose P faces a competing platform $\widetilde{\text{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1 \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\, 1, \phi_1\,)$. Proposition 2 implies that Sj ($j \in \{1, 2\}$) sells on P and faces no competition under PC. Lemma 2 and (10) imply that w_j is given by (174). (165), (169), and (174) imply that $w_j > w_j^M$ if $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1 \right)$.

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1} \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\, \phi_1, \phi_2\,)$. Proposition 2 implies that S1 competes against P whereas S2 sells on $\widetilde{\text{P}}$ and faces no competition under PC. Lemmas 2 and 5 imply that:

$$w_1 \;=\; \frac{1}{2\left[\, 1 - \Omega_1 \,\right]} \left\{ \frac{\Omega_1\, \overline{\Delta}_{P1}}{\eta_1} + \frac{\left[\, 8 + (\Omega_1)^2 \,\right]\, \Delta_{P1}}{\beta_1^S \left[\, 8 + \Omega_1 \,\right]} \right\} \;\text{ and }\; w_2 \;=\; \frac{\widetilde{\Delta}_{\widetilde{P}2}}{2\, b_2^S} \;=\; \frac{\widetilde{\Delta}_{\widetilde{P}2}}{2\, \beta_2^S \left[\, 1 - \Omega_2 \,\right]}. \tag{175}$$

The last equality in (175) reflects (10). (165) and (175) imply that $w_1 = w_1^M$. (172). (165), (173), and (175) imply that $w_2 > w_2^M$. ■

Proposition 8. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform $\widetilde{P}$ that is a weaker seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} > 1$). Then $SW < SW^M$ unless $\widetilde{P}$'s relative platform strength is sufficiently pronounced.*

Proof. (31), (33), and (108) imply that if S1 competes against Pk and S2 competes against Pi ($k, i \in \{1, 2\}$), then social welfare is:

$$\begin{aligned} SW \;&=\; \Pi_k + \Pi_i + \pi_1 + \pi_2 + CS \\ &=\; \left[\, p_{k1}^{*P} - c_{k1}^P \,\right] \Theta_k\, q_{k1}^{*P} - F + w_{k1}^*\, \Theta_k\, q_{k1}^{*S} + \left[\, p_{i2}^{*P} - c_{i2}^P \,\right] \Theta_i\, q_{i2}^{*P} - F + w_{i2}^*\, \Theta_i\, q_{i2}^{*S} \\ &\quad + \left[\, p_{k1}^{*S} - w_{k1}^* - c_1^S \,\right] \Theta_k\, q_{k1}^{*S} + \left[\, p_{i2}^{*S} - w_{i2}^* - c_2^S \,\right] \Theta_i\, q_{i2}^{*S} \\ &\quad + U\left(Q_{k1}^{*P}, Q_{i2}^{*P}, Q_{k1}^{*S}, Q_{i2}^{*S} \right) - p_{k1}^{*P}\, Q_{k1}^{*P} - p_{i2}^{*P}\, Q_{i2}^{*P} - p_{k1}^{*S}\, Q_{k1}^{*S} - p_{i2}^{*S}\, Q_{i2}^{*S} \end{aligned}$$

$$= \ -c_{k1}^{P}\,\Theta_k\,q_{k1}^{*P} - c_{i2}^{P}\,\Theta_i\,q_{i2}^{*P} - c_1^{S}\,\Theta_k\,q_{k1}^{*S} - c_2^{S}\,\Theta_i\,q_{i2}^{*S} + U\left(Q_{k1}^{*P}, Q_{i2}^{*P}, Q_{k1}^{*S}, Q_{i2}^{*S}\right) - 2\,F, \quad (176)$$

where q_{k1}^{*P}, q_{i2}^{*P}, q_{k1}^{*S}, and q_{i2}^{*S} are given by (113). (176) holds because $Q_{kj}^{*P} = \Theta_k\,q_{kj}^{*P}$ and $Q_{kj}^{*S} = \Theta_k\,q_{kj}^{*S}$. (109) imply that:

$$\begin{aligned}
U\left(Q_{k1}^{*P}, Q_{i2}^{*P}, Q_{k1}^{*S}, Q_{i2}^{*S}\right) \ &= \ \frac{\frac{1}{2}\,\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^S\,\frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} - \alpha_1\left[\,\eta_1 + \beta_1^S\,\theta_1\,\right]\,Q_{k1}^{*P}}{[\eta_1]^2 - \beta_1^S\,\beta_1^P} \\
&+ \ \frac{\frac{1}{2}\,\eta_2\,\frac{Q_{i2}^{*S}\,Q_{i2}^{*P}}{\Theta_i} + \frac{1}{2}\,\beta_2^S\,\frac{\left[Q_{i2}^{*P}\right]^2}{\Theta_i} - \alpha_2\left[\,\eta_2 + \beta_2^S\,\theta_2\,\right]\,Q_{i2}^{*P}}{[\eta_2]^2 - \beta_2^S\,\beta_2^P} \\
&+ \ \frac{\frac{1}{2}\,\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^P\,\frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k} - \alpha_1\left[\,\theta_1\,\eta_1 + \beta_1^P\,\right]\,Q_{k1}^{*S}}{[\eta_1]^2 - \beta_1^S\,\beta_1^P} \\
&+ \ \frac{\frac{1}{2}\,\eta_2\,\frac{Q_{i2}^{*P}\,Q_{i2}^{*S}}{\Theta_i} + \frac{1}{2}\,\beta_2^P\,\frac{\left[Q_{i2}^{*S}\right]^2}{\Theta_i} - \alpha_2\left[\,\theta_2\,\eta_2 + \beta_2^P\,\right]\,Q_{i2}^{*S}}{[\eta_2]^2 - \beta_2^S\,\beta_2^P}
\end{aligned}$$

$$\begin{aligned}
&= \ \frac{\eta_1\,\frac{Q_{k1}^{*S}\,Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^S\,\frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} + \frac{1}{2}\,\beta_1^P\,\frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k} - \alpha_1\left[\,\eta_1 + \beta_1^S\,\theta_1\,\right]\,Q_{k1}^{*P} - \alpha_1\left[\,\theta_1\,\eta_1 + \beta_1^P\,\right]\,Q_{k1}^{*S}}{[\eta_1]^2 - \beta_1^S\,\beta_1^P} \\
&+ \ \frac{\eta_2\,\frac{Q_{i2}^{*S}\,Q_{i2}^{*P}}{\Theta_i} + \frac{1}{2}\,\beta_2^S\,\frac{\left[Q_{i2}^{*P}\right]^2}{\Theta_i} - \alpha_2\left[\,\eta_2 + \beta_2^S\,\theta_2\,\right]\,Q_{i2}^{*P} + \frac{1}{2}\,\beta_2^P\,\frac{\left[Q_{i2}^{*S}\right]^2}{\Theta_i} - \alpha_2\left[\,\theta_2\,\eta_2 + \beta_2^P\,\right]\,Q_{i2}^{*S}}{[\eta_2]^2 - \beta_2^S\,\beta_2^P} \\
&= \ \Theta_k\,\frac{\eta_1\,q_{k1}^{*P}\,q_{k1}^{*S} + \frac{1}{2}\,\beta_1^S\left[q_{k1}^{*P}\right]^2 + \frac{1}{2}\,\beta_1^P\left[q_{k1}^{*S}\right]^2 - \alpha_1\left[\,\eta_1 + \beta_1^S\,\theta_1\,\right]\,q_{k1}^{*P} - \alpha_1\left[\,\theta_1\,\eta_1 + \beta_1^P\,\right]\,q_{k1}^{*S}}{[\eta_1]^2 - \beta_1^S\,\beta_1^P} \\
&+ \ \Theta_i\,\frac{\eta_2\,q_{i2}^{*P}\,q_{i2}^{*S} + \frac{1}{2}\,\beta_2^S\left[q_{i2}^{*P}\right]^2 + \frac{1}{2}\,\beta_2^P\left[q_{i2}^{*S}\right]^2 - \alpha_2\left[\,\eta_2 + \beta_2^S\,\theta_2\,\right]\,q_{i2}^{*P} - \alpha_2\left[\,\theta_2\,\eta_2 + \beta_2^P\,\right]\,q_{i2}^{*S}}{[\eta_2]^2 - \beta_2^S\,\beta_2^P} \\
&= \ \Theta_k\,\frac{\alpha_1\left[\,\eta_1 + \beta_1^S\,\theta_1\,\right]\,q_{k1}^{*P} + \alpha_1\left[\,\theta_1\,\eta_1 + \beta_1^P\,\right]\,q_{k1}^{*S} - \eta_1\,q_{k1}^{*P}\,q_{k1}^{*S} - \frac{1}{2}\,\beta_1^S\left[q_{k1}^{*P}\right]^2 - \frac{1}{2}\,\beta_1^P\left[q_{k1}^{*S}\right]^2}{\beta_1^S\,\beta_1^P\left[\,1 - \Omega_1\,\right]} \\
&+ \ \Theta_i\,\frac{\alpha_2\left[\,\eta_2 + \beta_2^S\,\theta_2\,\right]\,q_{i2}^{*P} + \alpha_2\left[\,\theta_2\,\eta_2 + \beta_2^P\,\right]\,q_{i2}^{*S} - \eta_2\,q_{i2}^{*P}\,q_{i2}^{*S} - \frac{1}{2}\,\beta_2^S\left[q_{i2}^{*P}\right]^2 - \frac{1}{2}\,\beta_2^P\left[q_{i2}^{*S}\right]^2}{\beta_2^S\,\beta_2^P\left[\,1 - \Omega_2\,\right]}.
\end{aligned} \quad (177)$$

(176) and (177) imply that if S1 competes against Pk and S2 competes against Pi ($k, i \in \{1, 2\}$), then social welfare is:

$$\begin{aligned}
SW \ &= \ -c_{k1}^{P}\,\Theta_k\,q_{k1}^{*P} - c_{i2}^{P}\,\Theta_i\,q_{i2}^{*P} - c_1^{S}\,\Theta_k\,q_{k1}^{*S} - c_2^{S}\,\Theta_i\,q_{i2}^{*S} - 2\,F \\
&+ \ \Theta_k\,\frac{\alpha_1\left[\,\eta_1 + \beta_1^S\,\theta_1\,\right]\,q_{k1}^{*P} + \alpha_1\left[\,\theta_1\,\eta_1 + \beta_1^P\,\right]\,q_{k1}^{*S} - \eta_1\,q_{k1}^{*P}\,q_{k1}^{*S} - \frac{1}{2}\,\beta_1^S\left[q_{k1}^{*P}\right]^2 - \frac{1}{2}\,\beta_1^P\left[q_{k1}^{*S}\right]^2}{\beta_1^S\,\beta_1^P\left[\,1 - \Omega_1\,\right]}
\end{aligned}$$

$$+ \;\; \Theta_i \, \frac{\alpha_2 \left[\, \eta_2 + \beta_2^S \, \theta_2 \, \right] \; q_{i2}^{*P} + \alpha_2 \left[\, \theta_2 \, \eta_2 + \beta_2^P \, \right] \; q_{i2}^{*S} - \eta_2 \, q_{i2}^{*P} \; q_{i2}^{*S} - \frac{1}{2} \, \beta_2^S \left[q_{i2}^{*P} \right]^2 - \frac{1}{2} \, \beta_2^P \left[q_{i2}^{*S} \right]^2}{\beta_2^S \, \beta_2^P \left[\, 1 - \Omega_2 \, \right]}$$

$$= \;\; \Theta_k \, q_{k1}^{*P} \left[\frac{\alpha_1 \left(\, \eta_1 + \beta_1^S \, \theta_1 \, \right)}{\beta_1^S \, \beta_1^P \left(\, 1 - \Omega_1 \, \right)} - c_{k1}^P \right] + \Theta_i \, q_{i2}^{*P} \left[\frac{\alpha_2 \left(\, \eta_2 + \beta_2^S \, \theta_2 \, \right)}{\beta_2^S \, \beta_2^P \left(\, 1 - \Omega_2 \, \right)} - c_{i2}^P \right]$$

$$+ \; \Theta_k \, q_{k1}^{*S} \left[\frac{\alpha_1 \left(\, \theta_1 \, \eta_1 + \beta_1^P \, \right)}{\beta_1^S \, \beta_1^P \left(\, 1 - \Omega_1 \, \right)} - c_1^S \right] + \Theta_i \, q_{i2}^{*S} \left[\frac{\alpha_2 \left(\, \theta_2 \, \eta_2 + \beta_2^P \, \right)}{\beta_2^S \, \beta_2^P \left(\, 1 - \Omega_2 \, \right)} - c_2^S \right] - 2 \, F$$

$$- \;\; \Theta_k \, \frac{\eta_1 \, q_{k1}^{*P} \; q_{k1}^{*S} + \frac{1}{2} \, \beta_1^S \left[q_{k1}^{*P} \right]^2 + \frac{1}{2} \, \beta_1^P \left[q_{k1}^{*S} \right]^2}{\beta_1^S \, \beta_1^P \left[\, 1 - \Omega_1 \, \right]} - \Theta_i \, \frac{\eta_2 \, q_{i2}^{*P} \; q_{i2}^{*S} + \frac{1}{2} \, \beta_2^S \left[q_{i2}^{*P} \right]^2 + \frac{1}{2} \, \beta_2^P \left[q_{i2}^{*S} \right]^2}{\beta_2^S \, \beta_2^P \left[\, 1 - \Omega_2 \, \right]} . \tag{178}$$

Observe that:

$$\frac{\alpha_j \left[\, \eta_j + \beta_j^S \, \theta_j \, \right]}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} - c_{kj}^P \;\; = \;\; \frac{\alpha_j \left[\, \eta_j + \beta_j^S \, \theta_j \, \right] - \beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right] c_{kj}^P}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]}$$

$$= \;\; \frac{\alpha_j \, \eta_j + \beta_j^S \, \theta_j \, \alpha_j - \beta_j^S \, \beta_j^P \, c_{kj}^P + \beta_j^S \, \beta_j^P \, \Omega_j \, c_{kj}^P}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]}$$

$$= \;\; \frac{\alpha_j \, \eta_j + \beta_j^S \left[\, \theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P \, \right] + \beta_j^S \, \beta_j^P \, \frac{\left[\eta_j \right]^2}{\beta_j^S \, \beta_j^P} \, c_{kj}^P}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} \;\; = \;\; \frac{\alpha_j \, \eta_j + \beta_j^S \left[\, \theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P \, \right] + \left[\eta_j \right]^2 \, c_{kj}^P}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]}$$

$$= \;\; \frac{\eta_j \left[\, \alpha_j + \eta_j \, c_{kj}^P \, \right] + \beta_j^S \left[\, \theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P \, \right]}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]}$$

$$= \;\; \frac{\eta_j \left[\, \alpha_j + \eta_j \, c_{kj}^P - \beta_j^S \, c_j^S \, \right] + \eta_j \, \beta_j^S \, c_j^S + \beta_j^S \left[\, \theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P \, \right]}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]}$$

$$= \;\; \frac{\eta_j \left[\, \alpha_j + \eta_j \, c_{kj}^P - \beta_j^S \, c_j^S \, \right] + \beta_j^S \left[\, \theta_j \, \alpha_j - \beta_j^P \, c_{kj}^P + \eta_j \, c_j^S \, \right]}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} \;\; = \;\; \frac{\eta_j \, \Delta_{kj} + \beta_j^S \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} ; \tag{179}$$

$$\frac{\alpha_j \left[\, \theta_j \, \eta_j + \beta_j^P \, \right]}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} - c_j^S \;\; = \;\; \frac{\alpha_j \left[\, \theta_j \, \eta_j + \beta_j^P \, \right] - \beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right] c_j^S}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} \;\; = \;\; \frac{\beta_j^P \, A_j - \beta_j^P \, b_j^S \, c_j^S}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} \tag{180}$$

$$= \;\; \frac{\widetilde{\Delta}_{kj}}{\beta_j^S \left[\, 1 - \Omega_j \, \right]} \;\; = \;\; \frac{\Delta_{kj} + \frac{\eta_j}{\beta_j^P} \, \overline{\Delta}_{kj}}{\beta_j^S \left[\, 1 - \Omega_j \, \right]} \;\; = \;\; \frac{\beta_j^P \, \Delta_{kj} + \eta_j \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \left[\, 1 - \Omega_j \, \right]} . \tag{181}$$

The last equality in (180) reflects (7) and (10). The first equality in (181) holds because $\widetilde{\Delta}_{kj} \equiv A_j - b_j^S \, c_j^S$. The second equality in (181) reflects (12).

(178), (179), and (180) imply that if S1 competes against Pk and S2 competes against

Pi $(k, i \in \{1,2\})$, then social welfare is:

$$SW = \Theta_k \, q_{k1}^{*P} \, \frac{\eta_1 \, \Delta_{k1} + \beta_1^S \, \overline{\Delta}_{k1}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} + \Theta_i \, q_{i2}^{*P} \, \frac{\eta_2 \, \Delta_{i2} + \beta_2^S \, \overline{\Delta}_{i2}}{\beta_2^S \, \beta_2^P \, [\, 1 - \Omega_2 \,]} - 2 \, F$$

$$+ \, \Theta_k \, q_{k1}^{*S} \, \frac{\beta_1^P \, \Delta_{k1} + \eta_1 \, \overline{\Delta}_{k1}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} + \Theta_i \, q_{i2}^{*S} \, \frac{\beta_2^P \, \Delta_{i2} + \eta_2 \, \overline{\Delta}_{i2}}{\beta_2^S \, \beta_2^P \, [\, 1 - \Omega_2 \,]}$$

$$- \, \Theta_k \, \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{1}{2} \, \beta_1^S \, \left[q_{k1}^{*P} \right]^2 + \frac{1}{2} \, \beta_1^P \, \left[q_{k1}^{*S} \right]^2}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} - \Theta_i \, \frac{\eta_2 \, q_{i2}^{*P} \, q_{i2}^{*S} + \frac{1}{2} \, \beta_2^S \, \left[q_{i2}^{*P} \right]^2 + \frac{1}{2} \, \beta_2^P \, \left[q_{i2}^{*S} \right]^2}{\beta_2^S \, \beta_2^P \, [\, 1 - \Omega_2 \,]} \tag{182}$$

where q_{k1}^{*P}, q_{i2}^{*P}, q_{k1}^{*S}, and q_{i2}^{*S} are given by (113).

(113), (115), and (182) imply that if S1 competes against Pk and S2 competes against Pi $(k, i \in \{1,2\})$, then social welfare is:

$$SW = \Theta_k \, \kappa_{k1} + \Theta_i \, \kappa_{i2} - 2 \, F, \tag{183}$$

where

$$\kappa_{kj} \equiv \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \frac{\eta_j \, \Delta_{kj} + \beta_j^S \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, \Delta_{kj}}{8 + \Omega_j} \frac{\beta_j^P \, \Delta_{kj} + \eta_j \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} - \varsigma_{kj}, \tag{184}$$

and ς_{kj} is given by (115).

(115) and (184) imply that:

$$\kappa_{kj} = \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \frac{\eta_j \, \Delta_{kj} + \beta_j^S \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, \Delta_{kj}}{8 + \Omega_j} \frac{\beta_j^P \, \Delta_{kj} + \eta_j \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]}$$

$$- \frac{1}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \left\{ \frac{\eta_j \, [\, 2 + \Omega_j \,] \, \Delta_{kj}}{8 + \Omega_j} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \right.$$

$$\left. + \frac{\beta_j^S}{2} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 + \frac{\beta_j^P}{2} \left[\frac{(\, 2 + \Omega_j \,) \, \Delta_{kj}}{8 + \Omega_j} \right]^2 \right\}$$

$$= \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right] \frac{\eta_j \, \Delta_{kj} + \beta_j^S \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, \Delta_{kj}}{8 + \Omega_j} \frac{\beta_j^P \, \Delta_{kj} + \eta_j \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]}$$

$$- \frac{\eta_j \, [\, 2 + \Omega_j \,] \, \Delta_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]$$

$$- \frac{1}{2 \, \beta_j^P \, [\, 1 - \Omega_j \,]} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j \, (\, 2 + \Omega_j \,) \, \Delta_{kj}}{2 \beta_j^S \, (\, 8 + \Omega_j \,)} \right]^2 - \frac{1}{2 \, \beta_j^S \, [\, 1 - \Omega_j \,]} \left[\frac{(\, 2 + \Omega_j \,) \, \Delta_{kj}}{8 + \Omega_j} \right]^2$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]\left[\eta_j\,\Delta_{kj}+\beta_j^S\,\overline{\Delta}_{kj}-\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]$$

$$+\frac{[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\frac{\beta_j^P\,\Delta_{kj}+\eta_j\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}$$

$$-\frac{1}{2\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]^2-\frac{1}{2\,\beta_j^S\,[\,1-\Omega_j\,]}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]^2 \quad (185)$$

Observe that:

$$\frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\right]\left[\eta_j\,\Delta_{kj}+\beta_j^S\,\overline{\Delta}_{kj}-\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\right]\left[\eta_j\,\Delta_{kj}\left(1-\frac{2+\Omega_j}{8+\Omega_j}\right)+\beta_j^S\,\overline{\Delta}_{kj}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\right]\left[\frac{6\,\eta_j\,\Delta_{kj}}{8+\Omega_j}+\beta_j^S\,\overline{\Delta}_{kj}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\overline{\Delta}_{kj}}{2}\left(\frac{6\,\eta_j\,\Delta_{kj}}{8+\Omega_j}+\beta_j^S\,\overline{\Delta}_{kj}\right)+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\,\beta_j^S\,(\,8+\Omega_j\,)}\left(\frac{6\,\eta_j\,\Delta_{kj}}{8+\Omega_j}+\beta_j^S\,\overline{\Delta}_{kj}\right)\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{3\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\beta_j^S\,\left(\overline{\Delta}_{kj}\right)^2}{2}+\frac{3\left(\eta_j\right)^2(\,2+\Omega_j\,)\;(\Delta_{kj})^2}{\beta_j^S\,(\,8+\Omega_j\,)^2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}\,\overline{\Delta}_{kj}}{2\,(\,8+\Omega_j\,)}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}\left[3+\frac{2+\Omega_j}{2}\right]+\frac{\beta_j^S\,\left(\overline{\Delta}_{kj}\right)^2}{2}+\frac{3\left(\eta_j\right)^2(\,2+\Omega_j\,)\;(\Delta_{kj})^2}{\beta_j^S\,(\,8+\Omega_j\,)^2}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}\,\frac{8+\Omega_j}{2}+\frac{\beta_j^S\,\left(\overline{\Delta}_{kj}\right)^2}{2}+\frac{3\left(\eta_j\right)^2(\,2+\Omega_j\,)\;(\Delta_{kj})^2}{\beta_j^S\,(\,8+\Omega_j\,)^2}\right]$$

$$= \frac{1}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}\left[\frac{\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{2}+\frac{\beta_j^S\,\left(\overline{\Delta}_{kj}\right)^2}{2}+\frac{3\left(\eta_j\right)^2(\,2+\Omega_j\,)\;(\Delta_{kj})^2}{\beta_j^S\,(\,8+\Omega_j\,)^2}\right]$$

$$= \frac{\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{2\,\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}+\frac{\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P\,[\,1-\Omega_j\,]}+\frac{3\,\Omega_j\,[\,2+\Omega_j\,]\;[\Delta_{kj}]^2}{\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]^2}; \quad (186)$$

$$\frac{[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\frac{\beta_j^P\,\Delta_{kj}+\eta_j\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]} = \frac{[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\frac{\beta_j^P\,\Delta_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}+\frac{[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\frac{\eta_j\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]}$$

$$= \frac{[\,2+\Omega_j\,]\;[\Delta_{kj}]^2}{\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]}+\frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]}; \quad (187)$$

$$
\begin{aligned}
& - \frac{1}{2\,\beta_j^P\,[\,1-\Omega_j\,]} \left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)} \right]^2 - \frac{1}{2\,\beta_j^S\,[\,1-\Omega_j\,]} \left[\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j} \right]^2 \\
= & - \frac{1}{8\,\beta_j^P\,[\,1-\Omega_j\,]} \left[\overline{\Delta}_{kj} + \frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)} \right]^2 - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \\
= & - \frac{1}{8\,\beta_j^P\,[\,1-\Omega_j\,]} \left[\left(\overline{\Delta}_{kj}\right)^2 + \frac{\left(\eta_j\right)^2 (\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{\left(\beta_j^S\right)^2 (\,8+\Omega_j\,)^2} + \frac{2\,\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)} \right] \\
& - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \\
= & - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{\left[\eta_j\right]^2 [\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]\,\left[\beta_j^S\right]^2 [\,8+\Omega_j\,]^2} \\
& - \frac{2\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[8+\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \\
= & - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \left[\frac{\left(\eta_j\right)^2}{4\,\beta_j^P\beta_j^S} + 1 \right] - \frac{2\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[\,8+\Omega_j\,]} \\
= & - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \left[\frac{\Omega_j}{4} + 1 \right] - \frac{2\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[\,8+\Omega_j\,]} \\
= & - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} \left[\frac{\Omega_j+4}{4} \right] - \frac{2\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[\,8+\Omega_j\,]} \\
= & - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\,[\,4+\Omega_j\,]\;[\Delta_{kj}]^2}{8\,\beta_j^S\,[\,1-\Omega_j\,]\,[8+\Omega_j\,]^2} - \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[\,8+\Omega_j\,]}. \qquad (188)
\end{aligned}
$$

(185) - (188) imply that:

$$
\begin{aligned}
\kappa_{kj} = & \frac{\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{2\,\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]} + \frac{\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P\,[\,1-\Omega_j\,]} + \frac{3\,\Omega_j\,[\,2+\Omega_j\,]\;[\Delta_{kj}]^2}{\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]^2} \\
& + \frac{[\,2+\Omega_j\,]\;[\Delta_{kj}]^2}{\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]} + \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]} \\
& - \frac{\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} - \frac{[\,2+\Omega_j\,]^2\,[\,4+\Omega_j\,]\;[\Delta_{kj}]^2}{8\,\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]^2} - \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^P\,[\,1-\Omega_j\,]\,\beta_j^S\,[\,8+\Omega_j\,]} \\
= & \frac{3\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} + \frac{[\,2+\Omega_j\,]\;[\Delta_{kj}]^2}{\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]} \left[1 + \frac{3\,\Omega_j}{8+\Omega_j} - \frac{(\,2+\Omega_j\,)\,(\,4+\Omega_j\,)}{8\,(\,8+\Omega_j\,)} \right]
\end{aligned}
$$

$$+ \frac{\eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \left[\frac{1}{2} + \frac{2 + \Omega_j}{8 + \Omega_j} - \frac{2 + \Omega_j}{4 \, (\, 8 + \Omega_j \,)} \right]$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \left[1 + \frac{3 \, \Omega_j}{8 + \Omega_j} - \frac{(\, 2 + \Omega_j \,) \, (\, 4 + \Omega_j \,)}{8 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \, \frac{2 \, [\, 8 + \Omega_j \,] + 3 \, [\, 2 + \Omega_j \,]}{4 \, [\, 8 + \Omega_j \,]}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \left[\frac{8 \, (\, 8 + \Omega_j \,) + 24 \, \Omega_j - (\, 2 + \Omega_j \,) \, (\, 4 + \Omega_j \,)}{8 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\eta_j \, \Delta_{kj} \, \overline{\Delta}_{kj}}{\beta_j^S \, \beta_j^P \, [\, 1 - \Omega_j \,]} \, \frac{22 + 5 \, \Omega_j}{4 \, [\, 8 + \Omega_j \,]}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \left[\frac{64 + 8 \, \Omega_j + 24 \, \Omega_j - \left[\, 8 + 6 \, \Omega_j + (\Omega_j)^2 \, \right]}{8 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{4 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,] \, [\, 1 - \Omega_j \,]}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \left[\frac{56 + 26 \, \Omega_j - (\Omega_j)^2}{8 \, (\, 8 + \Omega_j \,)} \right]$$

$$+ \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{4 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,] \, [\, 1 - \Omega_j \,]}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,] \, [\Delta_{kj}]^2}{\beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]} \, \frac{[\, 2 + \Omega_j \,] \, [\, 28 - \Omega_j \,]}{8 \, [\, 8 + \Omega_j \,]} + \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{4 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,] \, [\, 1 - \Omega_j \,]}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, [\Delta_{kj}]^2}{8 \, \beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]^2} + \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{4 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,] \, [\, 1 - \Omega_j \,]}. \tag{189}$$

(189) implies that:

$$\frac{\partial \kappa_{kj}}{\partial c_{kj}^P} = \frac{1}{4 \, [\, 1 - \Omega_j \,]} \left[\frac{6 \, \overline{\Delta}_{kj}}{2 \, \beta_j^P} \, \frac{\partial \overline{\Delta}_{kj}}{\partial c_{kj}^P} + \frac{(\, 2 + \Omega_j \,)^2 \, (\, 28 - \Omega_j \,) \, 2 \, \Delta_{kj}}{2 \, \beta_j^S \, (\, 8 + \Omega_j \,)^2} \, \frac{\partial \Delta_{kj}}{\partial c_{kj}^P} + \frac{\eta_j \, (\, 22 + 5 \, \Omega_j \,)}{\beta_j^S \, \beta_j^P \, (\, 8 + \Omega_j \,)} \, \frac{\partial \left(\Delta_{kj} \, \overline{\Delta}_{kj} \right)}{\partial c_{kj}^P} \right]$$

$$\stackrel{s}{=} -\frac{3 \, \overline{\Delta}_{kj}}{\beta_j^P} \, \beta_j^P + \frac{[\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, \Delta_{kj}}{\beta_j^S \, [\, 8 + \Omega_j \,]^2} \, \eta_j + \frac{\eta_j \, (\, 22 + 5 \, \Omega_j \,)}{\beta_j^S \, \beta_j^P \, (\, 8 + \Omega_j \,)} \left[\Delta_{kj} \, \frac{\partial \overline{\Delta}_{kj}}{\partial c_{kj}^P} + \overline{\Delta}_{kj} \, \frac{\partial \Delta_{kj}}{\partial c_{kj}^P} \right]$$

$$= -3 \, \overline{\Delta}_{kj} + \frac{\eta_j \, [\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, \Delta_{kj}}{\beta_j^S \, [\, 8 + \Omega_j \,]^2} + \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \left[\, -\Delta_{kj} \, \beta_j^P + \overline{\Delta}_{kj} \, \eta_j \, \right]}{\beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,]}$$

$$= \; -3\,\overline{\Delta}_{kj} + \frac{\eta_j\,[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,]\;\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]^2} - \frac{\eta_j\,[\,22+5\,\Omega_j\,]\;\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]} + \frac{\left[\eta_j\right]^2\,[\,22+5\,\Omega_j\,]\;\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,8+\Omega_j\,]}$$

$$= \; -3\,\overline{\Delta}_{kj} + \frac{\eta_j\,[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,]\;\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]^2} - \frac{\eta_j\,[\,22+5\,\Omega_j\,]\;\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]} + \frac{\Omega_j\,[\,22+5\,\Omega_j\,]\;\overline{\Delta}_{kj}}{[\,8+\Omega_j\,]}$$

$$= \; \overline{\Delta}_{kj}\left[\frac{\Omega_j\,(\,22+5\,\Omega_j\,)}{8+\Omega_j} - 3\right] + \frac{\eta_j\,\Delta_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}\left[\frac{(\,2+\Omega_j\,)^2\,(\,28-\Omega_j\,)}{8+\Omega_j} - (\,22+5\,\Omega_j\,)\right] \; < \; 0. \tag{190}$$

The inequality in (190) holds because

$$\frac{\Omega_j\,[\,22+5\,\Omega_j\,]}{8+\Omega_j} - 3 \; < \; 0 \quad \Leftrightarrow \quad 22\,\Omega_j + 5\,[\Omega_j]^2 - 24 - 3\,\Omega_j \; < \; 0 \quad \Leftrightarrow \quad 19\,\Omega_j + 5\,[\Omega_j]^2 - 24 \; < \; 0$$

$$\Leftrightarrow \quad [\,5\,\Omega_j + 24\,]\,[\,\Omega_j - 1\,] \; < \; 0, \text{ and} \tag{191}$$

$$\frac{[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,]}{8+\Omega_j} - [\,22+5\,\Omega_j\,] \; < \; 0 \quad \Leftrightarrow \quad [\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,] - [\,8+\Omega_j\,]\,[\,22+5\,\Omega_j\,] \; < \; 0. \tag{192}$$

(191) holds because $\Omega_j \in (0,1)$. (192) holds because $[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,] - [\,8+\Omega_j\,]\,[\,22+5\,\Omega_j\,]$ increases in $\Omega_j \in (0,1)$, and thus, $[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,] - [\,8+\Omega_j\,]\,[\,22+5\,\Omega_j\,] < \max\,[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,] - [\,8+\Omega_j\,]\,[\,22+5\,\Omega_j\,] = [\,2+1\,]^2\,[\,28-1\,] - [\,8+1\,]\,[\,22+5*1\,] = 0$.

Because $\frac{\widetilde{c}_j^P}{c_j^P} > 1$, (190) implies that:

$$\frac{\kappa_{Pj}}{\kappa_{\widetilde{P}j}} \; > \; 1. \tag{193}$$

(12) and (17) imply that:

$$\kappa_{kj} \; > \; \frac{7\left[\widetilde{\Delta}_{kj}\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]}$$

$$\Leftrightarrow \;\; \frac{3\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P\,[\,1-\Omega_j\,]} + \frac{[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,]\;[\Delta_{kj}]^2}{8\,\beta_j^S\,[\,1-\Omega_j\,]\,[\,8+\Omega_j\,]^2} + \frac{\eta_j\,[\,22+5\,\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,\beta_j^P\,[\,8+\Omega_j\,]\,[\,1-\Omega_j\,]}$$

$$> \; \frac{7\left[\,\Delta_{kj} + \frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\,\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]}$$

$$\Leftrightarrow \;\; \frac{3\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P} + \frac{[\,2+\Omega_j\,]^2\,[\,28-\Omega_j\,]\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,8+\Omega_j\,]^2} + \frac{\eta_j\,[\,22+5\,\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\,[\,8+\Omega_j\,]}$$

$$> \frac{7\left[\Delta_{kj}\right]^2 + \frac{7\left[\eta_j\right]^2\left[\overline{\Delta}_{kj}\right]^2}{\left[\beta_j^P\right]^2} + \frac{14\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^P}}{8\,\beta_j^S}$$

$$\Leftrightarrow \frac{3\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P} + \frac{\left[2+\Omega_j\right]^2\left[28-\Omega_j\right]\left[\Delta_{kj}\right]^2}{2\,\beta_j^S\left[8+\Omega_j\right]^2} + \frac{\eta_j\left[22+5\,\Omega_j\right]\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\left[8+\Omega_j\right]}$$

$$> \frac{7\left[\Delta_{kj}\right]^2}{8\,\beta_j^S} + \frac{7\left[\eta_j\right]^2\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^S\left[\beta_j^P\right]^2} + \frac{14\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{8\,\beta_j^S\,\beta_j^P}$$

$$\Leftrightarrow \frac{3\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P} + \frac{\left[2+\Omega_j\right]^2\left[28-\Omega_j\right]\left[\Delta_{kj}\right]^2}{2\,\beta_j^S\left[8+\Omega_j\right]^2} + \frac{\eta_j\left[22+5\,\Omega_j\right]\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,\beta_j^P\left[8+\Omega_j\right]}$$

$$> \frac{7\left[\Delta_{kj}\right]^2}{8\,\beta_j^S} + \frac{7\Omega_j\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P} + \frac{7\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,\beta_j^P}$$

$$\Leftrightarrow \frac{\left[12-7\Omega_j\right]\left[\overline{\Delta}_{kj}\right]^2}{8\,\beta_j^P} + \frac{4\left[2+\Omega_j\right]^2\left[28-\Omega_j\right]-7\left[8+\Omega_j\right]^2}{8\,\beta_j^S\left[8+\Omega_j\right]^2}\left[\Delta_{kj}\right]^2$$

$$+ \frac{4\,\eta_j\left[22+5\,\Omega_j\right]-7\,\eta_j\left[8+\Omega_j\right]}{4\,\beta_j^S\,\beta_j^P\left[8+\Omega_j\right]}\Delta_{kj}\,\overline{\Delta}_{kj} > 0. \qquad (194)$$

(194) holds because for $\Omega_j \in (0,1)$

$$\frac{4\left[2+\Omega_j\right]^2\left[28-\Omega_j\right]-7\left[8+\Omega_j\right]^2}{8\,\beta_j^S\left[8+\Omega_j\right]^2} > 0 \quad\Leftrightarrow\quad 4\left[2+\Omega_j\right]^2\left[28-\Omega_j\right] > 7\left[8+\Omega_j\right]^2$$

$$\Leftrightarrow\; 4\left[4+(\Omega_j)^2+4\,\Omega_j\right]\left[28-\Omega_j\right] > 7\left[64+(\Omega_j)^2+16\,\Omega_j\right]$$

$$\Leftrightarrow\; \left[4+(\Omega_j)^2+4\,\Omega_j\right]\left[112-4\,\Omega_j\right] > 448+7\left[\Omega_j\right]^2+112\,\Omega_j$$

$$\Leftrightarrow\; 4\left[112-4\,\Omega_j\right]+(\Omega_j)^2\left[112-4\,\Omega_j\right]+4\,\Omega_j\left[112-4\,\Omega_j\right] > 448+7\left[\Omega_j\right]^2+112\,\Omega_j$$

$$\Leftrightarrow\; 448-16\,\Omega_j+112\left[\Omega_j\right]^2-4\left[\Omega_j\right]^3+448\,\Omega_j-16\left[\Omega_j\right]^2 > 448+7\left[\Omega_j\right]^2+112\,\Omega_j$$

$$\Leftrightarrow\; 89\left[\Omega_j\right]^2-4\left[\Omega_j\right]^3+320\,\Omega_j > 0 \quad\Leftrightarrow\quad 89\,\Omega_j-4\left[\Omega_j\right]^2+320 > 0, \text{ and}$$

$$\frac{4\,\eta_j\left[22+5\,\Omega_j\right]-7\,\eta_j\left[8+\Omega_j\right]}{4\,\beta_j^S\,\beta_j^P\left[8+\Omega_j\right]} > 0 \quad\Leftrightarrow\quad 4\,\eta_j\left[22+5\,\Omega_j\right] > 7\,\eta_j\left[8+\Omega_j\right]$$

$$\Leftrightarrow\; 4\left[22+5\,\Omega_j\right] > 7\left[8+\Omega_j\right] \Leftrightarrow\; 88+20\,\Omega_j > 56+7\,\Omega_j \quad\Leftrightarrow\quad 32+13\,\Omega_j > 0\,.$$

(21), (26), and (123) imply that if S1 sells on Pk, S2 sells on Pi ($k, i \in \{1,2\}$), and each seller faces no competition, then social welfare is:

$$SW = \Pi_k + \Pi_i + \pi_1 + \pi_2 + CS$$

$$
\begin{aligned}
&= w^*_{k1}\,\Theta_k\,q^{*S}_{k1} + w^*_{i2}\,\Theta_i\,q^{*S}_{i2} + \left[\,p^{*S}_{k1} - w^*_{k1} - c^S_1\,\right]\Theta_k\,q^{*S}_{k1} + \left[\,p^{*S}_{i2} - w^*_{i2} - c^S_2\,\right]\Theta_i\,q^{*S}_{i2} \\
&\qquad + U\left(Q^{*S}_{k1}, Q^{*S}_{i2}\right) - p^{*S}_{k1}\,Q^{*S}_{k1} - p^{*S}_{i2}\,Q^{*S}_{i2} \\
&= -\,c^S_1\,\Theta_k\,q^{*S}_{k1} - c^S_2\,\Theta_i\,q^{*S}_{i2} + U\left(Q^{*S}_{k1}, Q^{*S}_{i2}\right),
\end{aligned}
\tag{195}
$$

where q^{*S}_{k1} and q^{*S}_{i2} are given by (128). (123) and (124) imply that:

$$
\begin{aligned}
U\left(Q^{*S}_{k1}, Q^{*S}_{i2}\right) &= \frac{A_1}{b^S_1}\,Q^{*S}_{k1} + \frac{A_2}{b^S_2}\,Q^{*S}_{i2} - \frac{1}{2}\left[\frac{\left(Q^{*S}_{k1}\right)^2}{b^S_1\,\Theta_k} + \frac{\left(Q^{*S}_{i2}\right)^2}{b^S_2\,\Theta_i}\right] \\
&= \frac{A_1}{b^S_1}\,\Theta_k\,q^{*S}_{k1} + \frac{A_2}{b^S_2}\,\Theta_i\,q^{*S}_{i2} - \frac{1}{2}\,\frac{\Theta_k\left(q^{*S}_{k1}\right)^2}{b^S_1} - \frac{1}{2}\,\frac{\Theta_i\left(q^{*S}_{i2}\right)^2}{b^S_2},
\end{aligned}
\tag{196}
$$

where q^{*S}_{k1} and q^{*S}_{i2} are given by (128). (195) and (196) imply that if S1 sells on Pk, S2 sells on Pi ($k, i \in \{1,2\}$), and each seller faces no competition, then social welfare is:

$$
SW = -\,c^S_1\,\Theta_k\,q^{*S}_{k1} - c^S_2\,\Theta_i\,q^{*S}_{i2} + \frac{A_1}{b^S_1}\,\Theta_k\,q^{*S}_{k1} + \frac{A_2}{b^S_2}\,\Theta_i\,q^{*S}_{i2} - \frac{1}{2}\,\frac{\Theta_k\left[q^{*S}_{k1}\right]^2}{b^S_1} - \frac{1}{2}\,\frac{\Theta_i\left[q^{*S}_{i2}\right]^2}{b^S_2}
$$

$$
= \Theta_k\,q^{*S}_{k1}\left[\frac{A_1}{b^S_1} - c^S_1\right] + \Theta_i\,q^{*S}_{i2}\left[\frac{A_2}{b^S_2} - c^S_2\right] - \frac{\Theta_k\left[q^{*S}_{k1}\right]^2}{2\,b^S_1} - \frac{\Theta_i\left[q^{*S}_{i2}\right]^2}{2\,b^S_2}
$$

$$
= \Theta_k\,q^{*S}_{k1}\left[\frac{A_1 - b^S_1\,c^S_1}{b^S_1}\right] + \Theta_i\,q^{*S}_{i2}\left[\frac{A_2 - b^S_2\,c^S_2}{b^S_2}\right] - \frac{\Theta_k\left[q^{*S}_{k1}\right]^2}{2\,b^S_1} - \frac{\Theta_i\left[q^{*S}_{i2}\right]^2}{2\,b^S_2}
$$

$$
= \Theta_k\,q^{*S}_{k1}\,\frac{\widetilde{\Delta}_{k1}}{\beta^S_1\left[\,1-\Omega_1\,\right]} + \Theta_i\,q^{*S}_{i2}\,\frac{\widetilde{\Delta}_{i2}}{\beta^S_2\left[\,1-\Omega_2\,\right]} - \frac{\Theta_k\left[q^{*S}_{k1}\right]^2}{2\,\beta^S_1\left[\,1-\Omega_1\,\right]} - \frac{\Theta_i\left[q^{*S}_{i2}\right]^2}{2\,\beta^S_2\left[\,1-\Omega_2\,\right]}
\tag{197}
$$

$$
= \Theta_k\,q^{*S}_{k1}\,\frac{2\,\widetilde{\Delta}_{k1} - q^{*S}_{k1}}{2\,\beta^S_1\left[\,1-\Omega_1\,\right]} + \Theta_i\,q^{*S}_{i2}\,\frac{2\,\widetilde{\Delta}_{i2} - q^{*S}_{i2}}{2\,\beta^S_2\left[\,1-\Omega_2\,\right]}
$$

$$
= \frac{\Theta_k\,\widetilde{\Delta}_{k1}\left[\,2\,\widetilde{\Delta}_{k1} - \frac{\widetilde{\Delta}_{k1}}{4}\right]}{8\,\beta^S_1\left[\,1-\Omega_1\,\right]} + \frac{\Theta_i\,\widetilde{\Delta}_{i2}\left[\,2\,\widetilde{\Delta}_{i2} - \frac{\widetilde{\Delta}_{i2}}{4}\right]}{8\,\beta^S_2\left[\,1-\Omega_2\,\right]}
\tag{198}
$$

$$
= \frac{\Theta_k\,\widetilde{\Delta}_{k1}\,7\,\widetilde{\Delta}_{k1}}{32\,\beta^S_1\left[\,1-\Omega_1\,\right]} + \frac{\Theta_i\,\widetilde{\Delta}_{i2}\,7\,\widetilde{\Delta}_{i2}}{32\,\beta^S_2\left[\,1-\Omega_2\,\right]} = \frac{7\,\Theta_k\left[\widetilde{\Delta}_{k1}\right]^2}{32\,\beta^S_1\left[\,1-\Omega_1\,\right]} + \frac{7\,\Theta_i\left[\widetilde{\Delta}_{i2}\right]^2}{32\,\beta^S_2\left[\,1-\Omega_2\,\right]}.
\tag{199}
$$

(197) holds because $\widetilde{\Delta}_{kj} \equiv A_j - b^S_j\,c^S_j$ and (10). (198) reflects (128).

(21), (26), (31), (33), and (130) imply that if S1 competes against Pk and S2 sells on Pi and faces no competition ($k, i \in \{1,2\}$), then social welfare is:

$$
\begin{aligned}
SW &= \Pi_k + \Pi_i + \pi_1 + \pi_2 + CS \\
&= \left[\,p^{*P}_{k1} - c^P_{k1}\,\right]\Theta_k\,q^{*P}_{k1} - F + w^*_{k1}\,\Theta_k\,q^{*S}_{k1} + w^*_{i2}\,\Theta_i\,q^{*S}_{i2} + \left[\,p^{*S}_{k1} - w^*_{k1} - c^S_1\,\right]\Theta_k\,q^{*S}_{k1} \\
&\quad + \left[\,p^{*S}_{i2} - w^*_{i2} - c^S_2\,\right]\Theta_i\,q^{*S}_{i2} + U\left(Q^{*P}_{k1}, Q^{*S}_{k1}, Q^{*S}_{i2}\right) - p^{*P}_{k1}\,Q^{*P}_{k1} - p^{*S}_{k1}\,Q^{*S}_{k1} - p^{*S}_{i2}\,Q^{*S}_{i2}
\end{aligned}
$$

$$= \; - c_{k1}^P \, \Theta_k \, q_{k1}^{*P} - c_1^S \, \Theta_k \, q_{k1}^{*S} - \Theta_i \, c_2^S \, q_{i2}^{*S} + U\left(Q_{k1}^{*P}, Q_{k1}^{*S}, Q_{i2}^{*S}\right) - F, \tag{200}$$

where q_{k1}^{*P} and q_{k1}^{*S} are given by (113) and q_{i2}^{*S} is given by (128). (130) and (131) imply that:

$$U\left(Q_{k1}^{*P}, Q_{k1}^{*S}, Q_{i2}^{*S}\right) \; = \; \frac{\frac{1}{2}\,\eta_1 \, \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^S \frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} - \alpha_1 \left[\, \eta_1 + \beta_1^S \, \theta_1 \,\right] \, Q_{k1}^{*P}}{[\eta_1]^2 - \beta_1^S \, \beta_1^P}$$

$$+ \; \frac{\frac{1}{2}\,\eta_1 \, \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} + \frac{1}{2}\,\beta_1^P \frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k} - \alpha_1 \left[\, \theta_1 \, \eta_1 + \beta_1^P \,\right] \, Q_{k1}^{*S}}{[\eta_1]^2 - \beta_1^S \, \beta_1^P} + \frac{A_2}{b_2^S}\, Q_{i2}^{*S} - \frac{\left[Q_{i2}^{*S}\right]^2}{2\, b_2^S \, \Theta_i}$$

$$= \; \frac{-\frac{1}{2}\,\eta_1 \, \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} - \frac{1}{2}\,\beta_1^S \frac{\left[Q_{k1}^{*P}\right]^2}{\Theta_k} + \alpha_1 \left[\, \eta_1 + \beta_1^S \, \theta_1 \,\right] \, Q_{k1}^{*P}}{\beta_1^S \, \beta_1^P - [\eta_1]^2}$$

$$+ \; \frac{-\frac{1}{2}\,\eta_1 \, \frac{Q_{k1}^{*S}\, Q_{k1}^{*P}}{\Theta_k} - \frac{1}{2}\,\beta_1^P \frac{\left[Q_{k1}^{*S}\right]^2}{\Theta_k} + \alpha_1 \left[\, \theta_1 \, \eta_1 + \beta_1^P \,\right] \, Q_{k1}^{*S}}{\beta_1^S \, \beta_1^P - [\eta_1]^2} + \frac{A_2}{b_2^S}\, Q_{i2}^{*S} - \frac{\left[Q_{i2}^{*S}\right]^2}{2\, b_2^S \, \Theta_i}$$

$$= \; \frac{-\frac{1}{2}\,\eta_1 \, \Theta_k \, q_{k1}^{*P} \, q_{k1}^{*S} - \frac{\beta_1^S \, \Theta_k}{2} \left[q_{k1}^{*P}\right]^2 + \alpha_1 \left[\, \eta_1 + \beta_1^S \, \theta_1 \,\right] \, \Theta_k \, q_{k1}^{*P}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]}$$

$$+ \; \frac{-\frac{1}{2}\,\eta_1 \, \Theta_k \, q_{k1}^{*P} \, q_{k1}^{*S} - \frac{\beta_1^P \, \Theta_k}{2} \left[q_{k1}^{*S}\right]^2 + \alpha_1 \left[\, \theta_1 \, \eta_1 + \beta_1^P \,\right] \Theta_k \, q_{k1}^{*S}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} + \frac{A_2}{b_2^S}\, \Theta_i \, q_{i2}^{*S} - \frac{\Theta_i \left[q_{i2}^{*S}\right]^2}{2\, b_2^S}$$

$$= \; \frac{-\eta_1 \, \Theta_k \, q_{k1}^{*P} \, q_{k1}^{*S} - \frac{\beta_1^S \, \Theta_k}{2} \left[q_{k1}^{*P}\right]^2 - \frac{\beta_1^P \, \Theta_k}{2} \left[q_{k1}^{*S}\right]^2 + \alpha_1 \left[\, \eta_1 + \beta_1^S \, \theta_1 \,\right] \, \Theta_k \, q_{k1}^{*P} + \alpha_1 \left[\, \theta_1 \, \eta_1 + \beta_1^P \,\right] \Theta_k \, q_{k1}^{*S}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]}$$

$$+ \; \frac{A_2}{b_2^S}\, \Theta_i \, q_{i2}^{*S} - \frac{\Theta_i \left[q_{i2}^{*S}\right]^2}{2\, b_2^S}, \tag{201}$$

where q_{k1}^{*P} and q_{k1}^{*S} are given by (113) and q_{i2}^{*S} is given by (128). (201) implies that (200) can be written as:

$$SW \; = \; -c_{k1}^P \, \Theta_k \, q_{k1}^{*P} - c_1^S \, \Theta_k \, q_{k1}^{*S} - \Theta_i \, c_2^S \, q_{i2}^{*S} - F + \frac{A_2}{b_2^S}\, \Theta_i \, q_{i2}^{*S} - \frac{\Theta_i \left[q_{i2}^{*S}\right]^2}{2\, b_2^S}$$

$$+ \; \frac{-\eta_1 \, \Theta_k \, q_{k1}^{*P} \, q_{k1}^{*S} - \frac{\beta_1^S \, \Theta_k}{2} \left[q_{k1}^{*P}\right]^2 - \frac{\beta_1^P \, \Theta_k}{2} \left[q_{k1}^{*S}\right]^2 + \alpha_1 \left[\, \eta_1 + \beta_1^S \, \theta_1 \,\right] \, \Theta_k \, q_{k1}^{*P} + \alpha_1 \left[\, \theta_1 \, \eta_1 + \beta_1^P \,\right] \Theta_k \, q_{k1}^{*S}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]}$$

$$= \; \Theta_k \, q_{k1}^{*P} \left[\frac{\alpha_1 \left(\, \eta_1 + \beta_1^S \, \theta_1 \,\right)}{\beta_1^S \, \beta_1^P \, (\, 1 - \Omega_1 \,)} - c_{k1}^P \right] + \Theta_k \, q_{k1}^{*S} \left[\frac{\alpha_1 \left(\, \theta_1 \, \eta_1 + \beta_1^P \,\right)}{\beta_1^S \, \beta_1^P \, (\, 1 - \Omega_1 \,)} - c_1^S \right] + \Theta_i \, q_{i2}^{*S} \left[\frac{A_2}{b_2^S} - c_2^S \right]$$

$$- \; \Theta_k \, \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{1}{2}\,\beta_1^S \left[q_{k1}^{*P}\right]^2 + \frac{1}{2}\,\beta_1^P \left[q_{k1}^{*S}\right]^2}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} - F - \frac{\Theta_i \left[q_{i2}^{*S}\right]^2}{2\, b_2^S}. \tag{202}$$

(179), (180) imply that (202) can be written as:

$$SW = \Theta_k \, q_{k1}^{*P} \, \frac{\eta_1 \, \Delta_{k1} + \beta_1^S \, \overline{\Delta}_{k1}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} + \Theta_k \, q_{k1}^{*S} \, \frac{\beta_1^P \, \Delta_{k1} + \eta_1 \, \overline{\Delta}_{k1}}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} + \Theta_i \, q_{i2}^{*S} \left[\frac{A_2 - b_2^S \, c_2^S}{b_2^S} \right]$$

$$- \, \Theta_k \, \frac{\eta_1 \, q_{k1}^{*P} \, q_{k1}^{*S} + \frac{1}{2} \, \beta_1^S \left[q_{k1}^{*P} \right]^2 + \frac{1}{2} \, \beta_1^P \left[q_{k1}^{*S} \right]^2}{\beta_1^S \, \beta_1^P \, [\, 1 - \Omega_1 \,]} - F - \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S}. \qquad (203)$$

(113), (128), and (115), (184), and (203) imply that if S1 competes against Pk and S2 sells on Pi and faces no competition ($k, i \in \{1, 2\}$), then social welfare is:

$$SW = \Theta_k \, \kappa_{k1} + \Theta_i \, q_{i2}^{*S} \left[\frac{A_2 - b_2^S \, c_2^S}{b_2^S} \right] - \frac{\Theta_i \left[q_{i2}^{*S} \right]^2}{2 \, b_2^S} - F$$

$$= \Theta_k \, \kappa_{k1} + \Theta_i \, \frac{\widetilde{\Delta}_{i2}}{4} \, \frac{\widetilde{\Delta}_{i2}}{\beta_2^S \, [\, 1 - \Omega_2 \,]} - \frac{\Theta_i \left[\frac{\widetilde{\Delta}_{i2}}{4} \right]^2}{2 \, \beta_2^S \, [\, 1 - \Omega_2 \,]} - F \qquad (204)$$

$$= \Theta_k \, \kappa_{k1} + \frac{\Theta_i \left[\widetilde{\Delta}_{i2} \right]^2}{4 \, \beta_2^S \, [\, 1 - \Omega_2 \,]} - \frac{\Theta_i \left[\widetilde{\Delta}_{i2} \right]^2}{32 \, \beta_2^S \, [\, 1 - \Omega_2 \,]} - F = \Theta_k \, \kappa_{k1} + \frac{7 \, \Theta_i \left[\widetilde{\Delta}_{i2} \right]^2}{32 \, \beta_2^S \, [\, 1 - \Omega_2 \,]} - F. \qquad (205)$$

(204) holds because $\widetilde{\Delta}_j \equiv A_j - b_j^S \, c_j^S$ and (10).

(16) and (189) imply that:

$$\kappa_{kj} - M_{kj} = \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{8 \, \beta_j^P \, [\, 1 - \Omega_j \,]} + \frac{[\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, [\Delta_{kj}]^2}{8 \, \beta_j^S \, [\, 1 - \Omega_j \,] \, [\, 8 + \Omega_j \,]^2} + \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{4 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,] \, [\, 1 - \Omega_j \,]}$$

$$- \, \frac{1}{2 \, [\, 1 - \Omega_j \,]} \left\{ \frac{\left(\overline{\Delta}_{kj} \right)^2}{2 \, \beta_j^P} + \frac{(\Delta_{kj})^2 \, (\, 2 + \Omega_j \,) \, \Omega_j}{2 \, \beta_j^S \, (\, 8 + \Omega_j \,)^2} \left[\, 26 - \Omega_j + 2 \, (\Omega_j)^2 \, \right] \right.$$

$$\left. + \, \frac{\Omega_j \, \overline{\Delta}_{kj} \, \Delta_{kj}}{\eta_j \, (\, 8 + \Omega_j \,)} \left[\, 6 + 2 \, \Omega_j + (\Omega_j)^2 \, \right] \right\}. \qquad (206)$$

(206) implies that:

$$2 \, [\, 1 - \Omega_j \,] \, [\, \kappa_{kj} - M_{kj} \,]$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{4 \, \beta_j^P} + \frac{[\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, [\Delta_{kj}]^2}{4 \, \beta_j^S \, [\, 8 + \Omega_j \,]^2} + \frac{\eta_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{2 \, \beta_j^S \, \beta_j^P \, [\, 8 + \Omega_j \,]}$$

$$- \left\{ \frac{\left(\overline{\Delta}_{kj} \right)^2}{2 \, \beta_j^P} + \frac{(\Delta_{kj})^2 \, (\, 2 + \Omega_j \,) \, \Omega_j}{2 \, \beta_j^S \, (\, 8 + \Omega_j \,)^2} \left[\, 26 - \Omega_j + 2 \, (\Omega_j)^2 \, \right] + \frac{\Omega_j \, \overline{\Delta}_{kj} \, \Delta_{kj}}{\eta_j \, (\, 8 + \Omega_j \,)} \left[\, 6 + 2 \, \Omega_j + (\Omega_j)^2 \, \right] \right\}$$

$$= \frac{3 \left[\overline{\Delta}_{kj} \right]^2}{4 \, \beta_j^P} + \frac{[\, 2 + \Omega_j \,]^2 \, [\, 28 - \Omega_j \,] \, [\Delta_{kj}]^2}{4 \, \beta_j^S \, [\, 8 + \Omega_j \,]^2} + \frac{\Omega_j \, [\, 22 + 5 \, \Omega_j \,] \, \Delta_{kj} \, \overline{\Delta}_{kj}}{2 \, \eta_j \, [\, 8 + \Omega_j \,]}$$

$$
\begin{aligned}
& - \frac{\left[\overline{\Delta}_{kj}\right]^2}{2\,\beta_j^P} - \frac{[\Delta_{kj}]^2\,[\,2+\Omega_j\,]\,\Omega_j}{2\,\beta_j^S\,[\,8+\Omega_j\,]^2}\left[\,26-\Omega_j+2\,(\Omega_j)^2\,\right] - \frac{\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{\eta_j\,[\,8+\Omega_j\,]}\left[\,6+2\,\Omega_j+(\Omega_j)^2\,\right] \\
= \; & \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{22+5\,\Omega_j-2\left[\,6+2\,\Omega_j+(\Omega_j)^2\,\right]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj} \\
& \quad + \frac{[\,2+\Omega_j\,]\;[\,28-\Omega_j\,]-2\,\Omega_j\left[\,26-\Omega_j+2\,(\Omega_j)^2\,\right]}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2 \\
= \; & \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{22+5\,\Omega_j-12-4\,\Omega_j-2\,[\Omega_j]^2}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj} \\
& \quad + \frac{56-[\Omega_j]^2+26\,\Omega_j-\left[\,52\,\Omega_j-2\,(\Omega_j)^2+4\,(\Omega_j)^3\,\right]}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2 \\
= \; & \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{10+\Omega_j-2\,[\Omega_j]^2}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj} + \frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2 \\
= \; & \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj} + \frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2 .
\end{aligned} \tag{207}
$$

Observe that:

$$
M_{kj} - \frac{\left[\,\widetilde{\Delta}_{kj}\,\right]^2}{8\,\beta_j^S\,[\,1-\Omega_j\,]} \;<\; \kappa_{kj} - \frac{7\left[\widetilde{\Delta}_{kj}\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]} \quad\Leftrightarrow\quad \frac{7\left[\widetilde{\Delta}_{kj}\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]} - \frac{\left[\,\widetilde{\Delta}_{kj}\,\right]^2}{8\,\beta_j^S\,[\,1-\Omega_j\,]} \;<\; \kappa_{kj} - M_{kj} \tag{208}
$$

$$
\Leftrightarrow \quad \frac{3\left[\widetilde{\Delta}_{kj}\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]} \;<\; \kappa_{kj}-M_{kj} \quad\Leftrightarrow\quad \frac{3\left[\widetilde{\Delta}_{kj}\right]^2}{16\,\beta_j^S} \;<\; 2\,[\,1-\Omega_j\,]\,[\,\kappa_{kj}-M_{kj}\,]\,. \tag{209}
$$

The last inequality in (209) holds because (207) implies that:

$$
\begin{aligned}
& \frac{3\left[\widetilde{\Delta}_{kj}\right]^2}{16\,\beta_j^S} \;<\; 2\,[\,1-\Omega_j\,]\,[\,\kappa_{kj}-M_{kj}\,] \\
\Leftrightarrow \quad & \frac{3\left[\widetilde{\Delta}_{kj}\right]^2}{16\,\beta_j^S} \;<\; \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj} \\
& \qquad\qquad\qquad + \frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2 \\
\Leftrightarrow \quad & \frac{3\left[\,\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\,\right]^2}{16\,\beta_j^S} \;<\; \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P} + \frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}
\end{aligned}
$$

$$+\ \frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2$$

$$\Leftrightarrow\quad \frac{3\,[\Delta_{kj}]^2+3\left[\frac{\eta_j}{\beta_j^P}\right]^2\ \left[\overline{\Delta}_{kj}\right]^2+\frac{6\,\eta_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{\beta_j^P}}{16\,\beta_j^S}$$

$$<\ \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P}+\frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}+\frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2$$

$$\Leftrightarrow\quad \frac{3\,[\Delta_{kj}]^2}{16\,\beta_j^S}+\frac{3\,\left[\eta_j\right]^2\ \left[\overline{\Delta}_{kj}\right]^2}{16\,\beta_j^S\,\left[\beta_j^P\right]^2}+\frac{6\,\eta_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{16\,\beta_j^S\,\beta_j^P}$$

$$<\ \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P}+\frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}+\frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2$$

$$\Leftrightarrow\quad \frac{3\,[\Delta_{kj}]^2}{16\,\beta_j^S}+\frac{3\,\left[\overline{\Delta}_{kj}\right]^2}{16\,\beta_j^P}+\frac{3\,\eta_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{8\,\beta_j^S\,\beta_j^P}$$

$$<\ \frac{\left[\overline{\Delta}_{kj}\right]^2}{4\,\beta_j^P}+\frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}+\frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2$$

$$\Leftrightarrow\quad \frac{\left[\overline{\Delta}_{kj}\right]^2}{16\,\beta_j^P}+\frac{[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]}{2\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}-\frac{3\,\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}}{8\,\eta_j}$$

$$+\ \frac{56-26\,\Omega_j+[\Omega_j]^2-4\,[\Omega_j]^3}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]\,[\Delta_{kj}]^2-\frac{3\,[\Delta_{kj}]^2}{16\,\beta_j^S}\ >\ 0$$

$$\Leftrightarrow\quad \frac{\left[\overline{\Delta}_{kj}\right]^2}{16\,\beta_j^P}+\frac{4\,[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]-3\,[\,8+\Omega_j\,]}{8\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}$$

$$+\ \frac{4\left[\,56-26\,\Omega_j+(\Omega_j)^2-4\,(\Omega_j)^3\,\right][\,2+\Omega_j\,]-3\,[\,8+\Omega_j\,]^2}{16\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\Delta_{kj}]^2\ >\ 0. \tag{210}$$

Observe that:

$$4\,[\,2+\Omega_j\,]\,[\,5-2\,\Omega_j\,]-3\,[\,8+\Omega_j\,]\ =\ [\,8+4\,\Omega_j\,]\,[\,5-2\,\Omega_j\,]-24-3\,\Omega_j$$

$$=\ 40-8\,[\Omega_j]^2+4\,\Omega_j-24-3\,\Omega_j\ =\ 16-8\,[\Omega_j]^2+\Omega_j; \tag{211}$$

$$4\left[\,56-26\,\Omega_j+(\Omega_j)^2-4\,(\Omega_j)^3\,\right][\,2+\Omega_j\,]-3\,[\,8+\Omega_j\,]^2$$

$$=\ \left[\,56-26\,\Omega_j+(\Omega_j)^2-4\,(\Omega_j)^3\,\right][\,8+4\,\Omega_j\,]-3\,[\,8+\Omega_j\,]^2$$

$$=\ 8\left[\,56-26\,\Omega_j+(\Omega_j)^2-4\,(\Omega_j)^3\,\right]+4\,\Omega_j\left[\,56-26\,\Omega_j+(\Omega_j)^2-4\,(\Omega_j)^3\,\right]-3\,[\,8+\Omega_j\,]^2$$

$$=\ 448-208\,\Omega_j+8\,[\Omega_j]^2-32\,[\Omega_j]^3+224\,\Omega_j-104\,[\Omega_j]^2+4\,[\Omega_j]^3-16\,[\Omega_j]^4-3\,[\,8+\Omega_j\,]^2$$

$$= \; 448 + 16\,\Omega_j - 96\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 - 3\left[\, 64 + (\Omega_j)^2 + 16\,\Omega_j \,\right]$$

$$= \; 448 + 16\,\Omega_j - 96\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 - 192 - 3\,[\Omega_j]^2 - 48\,\Omega_j$$

$$= \; 256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 \,. \tag{212}$$

(210) - (212) imply that:

$$\frac{3\left[\widetilde{\Delta}_{kj}\right]^2}{16\,\beta_j^S} \;<\; 2\,[\,1-\Omega_j\,]\,[\,\kappa_{kj} - M_{kj}\,]$$

$$\Leftrightarrow \quad \frac{\left[\overline{\Delta}_{kj}\right]^2}{16\,\beta_j^P} + \frac{16 - 8\,[\Omega_j]^2 + \Omega_j}{8\,\eta_j\,[\,8+\Omega_j\,]}\Omega_j\,\overline{\Delta}_{kj}\,\Delta_{kj}$$

$$+ \; \frac{256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4}{16\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\Delta_{kj}]^2 \;\;>\;\; 0. \tag{213}$$

(213) holds because for $\Omega_j \in (0,1)$

$$16 - 8\,[\Omega_j]^2 + \Omega_j \;\;>\;\; 0 \;\;\text{and} \tag{214}$$

$$256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 \;\;>\;\; 0. \tag{215}$$

(214) holds because $16 - 8\,[\Omega_j]^2 + \Omega_j$ decreases in $\Omega_j \in (0,1)$, and thus, $16 - 8\,[\Omega_j]^2 + \Omega_j > \min 16 - 8\,[\Omega_j]^2 + \Omega_j = 16 - 8\,[1]^2 + 1 = 9 > 0$. (215) holds because $256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4$ decreases in $\Omega_j \in (0,1)$, and thus, $256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 > \min 256 - 32\,\Omega_j - 99\,[\Omega_j]^2 - 28\,[\Omega_j]^3 - 16\,[\Omega_j]^4 = 256 - 32 * 1 - 99\,[1]^2 - 28\,[1]^3 - 16\,[1]^4 = 81 > 0$.

(208) implies that:

$$M_{P1} - \frac{\left[\widetilde{\Delta}_{P1}\right]^2}{8\,\beta_1^S\,[\,1-\Omega_1\,]} \;\;<\;\; \kappa_{P1} - \frac{7\left[\widetilde{\Delta}_{P1}\right]^2}{64\,\beta_1^S\,[\,1-\Omega_1\,]}; \tag{216}$$

$$\Theta\,M_{P2} - \frac{\Theta\left[\,\widetilde{\Delta}_{P2}\,\right]^2}{8\,\beta_2^S\,[\,1-\Omega_2\,]} \;\;<\;\; \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{217}$$

Condition FS ensures that $F < \Theta\,M_{P1} - \frac{\Theta\left[\,\widetilde{\Delta}_{P1}\,\right]^2}{8\,b_1^S} = \Theta\,M_{P1} - \frac{\Theta\left[\,\widetilde{\Delta}_{P1}\,\right]^2}{8\,\beta_1^S\,[1-\Omega_1\,]}$ and $F < \Theta\,M_{P2} - \frac{\Theta\left[\,\widetilde{\Delta}_{P2}\,\right]^2}{8\,b_2^S} = \Theta\,M_{P2} - \frac{\Theta\left[\,\widetilde{\Delta}_{P2}\,\right]^2}{8\,\beta_2^S\,[1-\Omega_2\,]}$. Therefore, (216) and (217) imply that:

$$F \;\;<\;\; \Theta\,\kappa_{P1} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{64\,\beta_1^S\,[\,1-\Omega_1\,]}; \tag{218}$$

$$F \; < \; \Theta\, \kappa_{P2} - \frac{7\, \Theta \left[\widetilde{\Delta}_{P2} \right]^2}{32\, \beta_2^S \left[\, 1 - \Omega_2 \,\right]}. \tag{219}$$

Proposition 1 implies that each seller competes against P under MP. Therefore, (183) implies that:

$$SW^M \; = \; \Theta\, \kappa_{P1} + \Theta\, \kappa_{P2} - 2\, F, \tag{220}$$

where κ_{kj} is given by (189).

First suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} > \max \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}, \phi_{\widetilde{P}2} \right\}$.

Because $\frac{\widetilde{\Theta}}{\Theta} > \phi_{\widetilde{P}2}$, Proposition 2 implies that each seller competes against $\widetilde{\mathrm{P}}$ under PC. Therefore, (183) implies that:

$$SW \; = \; \widetilde{\Theta}\, \kappa_{\widetilde{P}1} + \widetilde{\Theta}\, \kappa_{\widetilde{P}2} - 2\, F, \tag{221}$$

where κ_{kj} is given by (189).

Because $\frac{\widetilde{\Theta}}{\Theta} > \max \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}} \right\}$, $\frac{\widetilde{\Theta}}{\Theta} > \frac{\kappa_{Pj}}{\kappa_{\widetilde{P}j}}$ ($j \in \{1, 2\}$), and thus, $\widetilde{\Theta}\, \kappa_{\widetilde{P}j} > \Theta\, \kappa_{Pj}$. Therefore, (220) and (221) imply that $SW > SW^M$ in this case.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \min \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}, \phi_{\widetilde{P}2} \right\} \right)$.

First suppose $\phi_{\widetilde{P}1} < \min \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}} \right\}$.

If $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \min \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}, \phi_{\widetilde{P}2} \right\} \right)$, then $\frac{\widetilde{\Theta}}{\Theta} \in \left(\phi_{\widetilde{P}1}, \phi_{\widetilde{P}2} \right)$. Proposition 2 implies that S1 competes against $\widetilde{\mathrm{P}}$ whereas S2 sells on P and faces no competition under PC. Therefore, (205) implies that:

$$SW \; = \; \widetilde{\Theta}\, \kappa_{\widetilde{P}1} + \frac{7\, \Theta \left[\widetilde{\Delta}_{P2} \right]^2}{32\, \beta_2^S \left[\, 1 - \Omega_2 \,\right]} - F. \tag{222}$$

Because $\frac{\widetilde{\Theta}}{\Theta} < \min \left\{ \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}} \right\}$, $\frac{\widetilde{\Theta}}{\Theta} < \frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}$, and thus,

$$\widetilde{\Theta}\, \kappa_{\widetilde{P}1} \; < \; \Theta\, \kappa_{P1}. \tag{223}$$

(194) implies that $\frac{7 \left[\widetilde{\Delta}_{P2} \right]^2}{32\, \beta_2^S \left[1 - \Omega_2 \right]} < \kappa_{P2}$. Therefore, (140) and (143) imply that in this case:

$$SW \; < \; SW^M \quad \Leftrightarrow \quad \widetilde{\Theta}\, \kappa_{\widetilde{P}1} + \frac{7\, \Theta \left[\widetilde{\Delta}_{P2} \right]^2}{32\, \beta_2^S \left[\, 1 - \Omega_2 \,\right]} - F \; < \; \Theta\, \kappa_{P1} + \Theta\, \kappa_{P2} - 2\, F$$

$$\Leftrightarrow \;\; F \;<\; \Theta\,\kappa_{P1} - \widetilde{\Theta}\,\kappa_{\widetilde{P}1} + \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{224}$$

(224) holds because (219) and (223) imply that $\Theta\,\kappa_{P1} - \widetilde{\Theta}\,\kappa_{\widetilde{P}1} > 0$, and thus,

$$F \;<\; \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} \;<\; \Theta\,\kappa_{P1} - \widetilde{\Theta}\,\kappa_{\widetilde{P}1} + \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}.$$

If $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$, Proposition 2 implies that each seller sells on $\widetilde{\mathrm{P}}$ and faces no competition under PC. Therefore, (199) implies that:

$$SW \;=\; \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} + \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{225}$$

(194) implies that $\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{32\,\beta_j^S[\,1-\Omega_j\,]} < \widetilde{\Theta}\,\kappa_{\widetilde{P}j}$. Because $\frac{\widetilde{\Theta}}{\Theta} < \phi_{\widetilde{P}1} < \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}\right\}$, $\frac{\widetilde{\Theta}}{\Theta} < \frac{\kappa_{Pj}}{\kappa_{\widetilde{P}j}}$ ($j \in \{1,2\}$), and thus,

$$\widetilde{\Theta}\,\kappa_{\widetilde{P}j} \;<\; \Theta\,\kappa_{Pj}. \tag{226}$$

Therefore, $\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}j}\right]^2}{32\,\beta_j^S[\,1-\Omega_j\,]} < \Theta\,\kappa_{Pj}$. Condition FS ensures that $F < \widetilde{\Theta}\,M_{\widetilde{P}2} - \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{8\,b_2^S} = \widetilde{\Theta}\,M_{\widetilde{P}2} - \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{8\,\beta_2^S[\,1-\Omega_2\,]}$ and $F < \widetilde{\Theta}\,M_{\widetilde{P}1} - \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{8\,b_1^S} = \widetilde{\Theta}\,M_{\widetilde{P}1} - \frac{\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{8\,\beta_1^S[\,1-\Omega_1\,]}$. Therefore, (208) implies that

$$F \;<\; \widetilde{\Theta}\,\kappa_{\widetilde{P}1} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} \;\;\text{and}\;\; F \;<\; \widetilde{\Theta}\,\kappa_{\widetilde{P}2} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{227}$$

(226) and (227) imply that:

$$F \;<\; \Theta\,\kappa_{P1} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} \;\;\text{and}\;\; F \;<\; \Theta\,\kappa_{P2} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{228}$$

Therefore, (220) and (225) imply that in this case:

$$SW \;<\; SW^M \;\;\Leftrightarrow\;\; \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} + \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} \;<\; \Theta\,\kappa_{P1} + \Theta\,\kappa_{P2} - 2\,F$$

$$\Leftrightarrow \;\; 2\,F \;<\; \Theta\,\kappa_{P1} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} + \Theta\,\kappa_{P2} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{229}$$

(228) implies that (229) holds.

Next suppose $\phi_{\widetilde{P}1} \geq \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}\right\}$. (91) implies that $\phi_{\widetilde{P}2} > \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}\right\}$. Therefore, $\min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\} = \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}\right\}$. Therefore, $\frac{\widetilde{\Theta}}{\Theta} < \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}, \phi_{\widetilde{P}2}\right\} = \min\left\{\frac{\kappa_{P1}}{\kappa_{\widetilde{P}1}}, \frac{\kappa_{P2}}{\kappa_{\widetilde{P}2}}\right\} \leq \phi_{\widetilde{P}1}$. Because $\frac{\widetilde{\Theta}}{\Theta} > 1$ in this case, then $\frac{\widetilde{\Theta}}{\Theta} \in \left(1, \phi_{\widetilde{P}1}\right)$. Proposition 2 implies that each seller sells on $\widetilde{\text{P}}$ and faces no competition under PC. Therefore, consumer surplus is given by (225). (220), (225), and (229) imply that $SW < SW^M$ in this case.

Next suppose P faces a symmetric competing platform $\widetilde{\text{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 2 implies that each seller is indifferent between selling on P and selling on $\widetilde{\text{P}}$ and each seller faces no competition under PC. Therefore, (199) implies that:

$$\begin{aligned} SW \;&=\; \frac{1}{2}\,\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{1}{2}\,\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]} + \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]} \\ &=\; \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]} + \frac{1}{2}\,\frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]} \qquad (230) \\ &=\; \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]}. \qquad (231) \end{aligned}$$

(230) holds because $\frac{\widetilde{\Theta}}{\Theta} = 1$ and $\widetilde{\Delta}_{\widetilde{P}j} = \widetilde{\Delta}_{Pj}$. Therefore, (220) and (231) imply that in this case:

$$\begin{aligned} SW \;<\; SW^M \;&\Leftrightarrow\; \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]} \;<\; \Theta\,\kappa_{P1} + \Theta\,\kappa_{P2} - 2\,F \\ &\Leftrightarrow\; 2\,F \;<\; \Theta\,\kappa_{P1} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]}. \qquad (232) \end{aligned}$$

(218) and (219) imply that (232) holds.

Finally, suppose P faces a competing platform $\widetilde{\text{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

<u>Case I</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P1}}, 1\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (1, \phi_{P1})$. Proposition 2 implies that each seller sells on P and faces no competition under PC. Therefore, (199) implies that:

$$SW \;=\; \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\left[1-\Omega_1\right]} + \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\left[1-\Omega_2\right]}. \qquad (233)$$

(194) implies that $\frac{7\left[\widetilde{\Delta}_{Pj}\right]^2}{32\,\beta_j^S\,[\,1-\Omega_j\,]} < \kappa_{Pj}$ for $j \in \{1,2\}$. Therefore, (220) and (233) imply that:

$$SW \;<\; SW^M \;\;\Leftrightarrow\;\; \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} + \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} \;<\; \Theta\,\kappa_{P1} + \Theta\,\kappa_{P2} - 2\,F$$

$$\Leftrightarrow\;\; 2\,F \;<\; \Theta\,\kappa_{P1} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P1}\right]^2}{32\,\beta_1^S\,[\,1-\Omega_1\,]} + \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{234}$$

(218) and (219) imply that (234) holds.

<u>Case II</u>. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_{P2}}, \frac{1}{\phi_{P1}}\right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\,\phi_{P1}, \phi_{P2}\,)$. Proposition 2 implies that S1 competes against P whereas S2 sells on $\widetilde{\text{P}}$ and faces no competition under PC. Therefore, (205) implies that:

$$SW \;=\; \Theta\,\kappa_{P1} + \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} - F. \tag{235}$$

(194) implies that $\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} < \widetilde{\Theta}\,\kappa_{\widetilde{P}2}$. (193) and $\frac{\widetilde{\Theta}}{\Theta} < 1$ imply that $\frac{\widetilde{\Theta}}{\Theta} < \frac{\kappa_{Pj}}{\kappa_{\widetilde{P}j}}$, and thus, $\widetilde{\Theta}\,\kappa_{\widetilde{P}j} < \Theta\,\kappa_{Pj}$. Therefore, $\frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} < \Theta\,\kappa_{Pj}$. Therefore, (220) and (235) imply that in this case:

$$SW \;<\; SW^M \;\;\Leftrightarrow\;\; \Theta\,\kappa_{P1} + \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} - F \;<\; \Theta\,\kappa_{P1} + \Theta\,\kappa_{P2} - 2\,F$$

$$\Leftrightarrow\;\; F \;<\; \Theta\,\kappa_{P2} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}. \tag{236}$$

Because $\frac{\widetilde{\Theta}}{\Theta} < 1$ in this case and $\widetilde{\Delta}_{\widetilde{P}2} = \widetilde{\Delta}_{P2}$ (since $\frac{\partial\widetilde{\Delta}_{kj}}{\partial c_{kj}^P} = 0$), then $\Theta\,\kappa_{P2} - \frac{7\,\widetilde{\Theta}\left[\widetilde{\Delta}_{\widetilde{P}2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]} > \Theta\,\kappa_{P2} - \frac{7\,\Theta\left[\widetilde{\Delta}_{P2}\right]^2}{32\,\beta_2^S\,[\,1-\Omega_2\,]}$. Therefore, (219) implies that (236) must hold. ■

Proposition 9. *Suppose Condition FS holds, P's selling strength relative to its rival seller (Sj) is sufficiently pronounced (i.e.,* $\frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} > \frac{-\overline{b}+\sqrt{\left(\overline{b}\right)^2-4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}$*), and* $\widetilde{P}$ *is sufficiently similar to P (i.e.,* $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{16[\,1-\Omega_j\,]\left[\,\widetilde{\Delta}_j+\Delta_{Pj}\,\right]^2}{[\,4-\Omega_j\,]^2\left[\widetilde{\Delta}_j\right]^2}, \min\left\{\frac{\varsigma_{Pj}}{\frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2}, \frac{[\,4-\Omega_j\,]^2\left[\widetilde{\Delta}_j\right]^2}{16[\,1-\Omega_j\,]\left[\,\widetilde{\Delta}_j+\Delta_{\widetilde{P}j}\,\right]^2}\right\}\right)$*).*[4] *Further suppose*

[4] $\overline{a}$, $\overline{b}$, and $\overline{c}$ are as specified in (20).

platforms can make ex ante long-term commitments regarding both commissions and their selling capabilities. Then $CS < CS^M$.

<u>Proof</u>**.** The proof will proceed in three steps. First, I show that under MP, both sellers sell on P, P does not make a no-entry commitment, and P charges its profit-maximizing commissions in equilibrium. Next, I show that under PC, if platforms have similar platform strengths, then no platform has an incentive to solely employ the commission instrument. Therefore, each platform either solely commits not to enter or employs both instruments in equilibrium under PC. Finally, I compare consumer surplus under MP and under PC.

Under MP, each seller competes against P, P does not make a no-entry commitment, and P charges its profit-maximizing commissions in equilibrium. This is the case because each seller secures zero profit if it does not sell on P. Therefore, each seller sells on P that competes against sellers and charges its profit-maximizing commissions in equilibrium.[5] (140) implies that consumer surplus under MP is:

$$CS^M \;=\; \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}, \tag{237}$$

Now suppose P faces a competing platform $\widetilde{\mathrm{P}}$ under PC.

Lemma 3 implies that if Sj sells on Pk and Pk solely employs the no-entry instrument, then Sj's profit is:

$$\pi^N_{kj} \;=\; \frac{\Theta_k \left[\, \widetilde{\Delta}_{kj} \,\right]^2}{16\, b^S_j}, \tag{238}$$

and Pk's profit from the commission it collects from Sj is

$$\Pi^N_{kj} \;=\; \frac{\Theta_k \left[\, \widetilde{\Delta}_{kj} \,\right]^2}{8\, b^S_j}. \tag{239}$$

Lemma 4 implies that if Sj sells on Pk and Pk solely employs the commission instrument, then Sj's profit is:

$$\pi^C_{kj}\,(w_{kj}) \;=\; \frac{\Theta_k \left[\, \frac{\eta_j}{\beta^P_j}\, \overline{\Delta}_{kj} + 2\,\Delta_{kj} - 2\,\beta^S_j\,(\,1-\Omega_j\,)\;w_{kj} \,\right]^2}{\beta^S_j\,[\,4-\Omega_j\,]^2} \;=\; \frac{\Theta_k \left[\, \widetilde{\Delta}_{kj} + \Delta_{kj} - 2\,b^S_j\,w_{kj} \,\right]^2}{\beta^S_j\,[\,4-\Omega_j\,]^2}, \tag{240}$$

and Pk's profit, which is the sum of its commission revenue from Sj and its retail revenue, is:

$$\Pi^C_{kj}\,(w_{kj}) \;=\; \left[\, p^P_{kj}\,(w_{kj}) - c^P_{kj} \,\right] Q^P_{kj}\,(w_{kj}) - F + w_{kj}\,Q^S_{kj}\,(w_{kj})\,,$$

where $w^{01}_{kj} \geq 0$ is the commission to which Pk commits before sellers make their platform

[5] Proposition 1 has shown that when the no-entry instrument is available, P has no incentive to employ it under MP. In addition, as the monopolistic platform, P is indifferent between committing to the profit-maximizing commission *ex ante* and setting it *ex post*.

choices.[6] The last equality in (240) reflects (10) and (12).

Lemma 1 implies that if Sj sells on Pk and Pk employs both instruments, then Sj's profit is:

$$\pi_{kj}^{B}\left(w_{kj}\right) \;=\; \frac{\Theta_k\left[\,\widetilde{\Delta}_{kj}-b_j^S\,w_{kj}\,\right]^2}{4\,b_j^S} \;=\; \frac{\Theta_k\left[\,\widetilde{\Delta}_{kj}-b_j^S\,w_{kj}\,\right]^2}{4\,\beta_j^S\left[\,1-\Omega_j\,\right]}, \tag{241}$$

and Pk's profit from the commission it collects from Sj is:

$$\Pi_{kj}^{B}\left(w_{kj}\right) \;=\; w_{kj}\,Q_{kj}^S \;=\; w_{kj}\,\frac{\Theta_k\left[\,\widetilde{\Delta}_{kj}-b_j^S\,w_{kj}\,\right]}{2}, \tag{242}$$

where $w_{kj} \geq 0$ is the commission to which Pk commits before sellers make their platform choices. The last equality in (241) reflects (10).

(12) implies that $\frac{\partial \widetilde{\Delta}_{kj}}{\partial c_{kj}^P} = 0$. Therefore, for $k, i, j \in \{1,2\}$ and $k \neq i$:

$$\widetilde{\Delta}_{ij} \;=\; \widetilde{\Delta}_{kj} \;=\; \widetilde{\Delta}_j. \tag{243}$$

Because P and $\widetilde{\text{P}}$ have similar platform strengths, i.e.,

$$\frac{\widetilde{\Theta}}{\Theta} \;\in\; \left(\frac{16\left[\,1-\Omega_j\,\right]\left[\,\widetilde{\Delta}_j+\Delta_{Pj}\,\right]^2}{\left[\,4-\Omega_j\,\right]^2\left[\,\widetilde{\Delta}_j\,\right]^2},\ \min\left\{\frac{\varsigma_{Pj}}{\frac{1}{2\,b_j^S}\left[\,\frac{\widetilde{\Delta}_j}{2}\,\right]^2},\ \frac{\left[\,4-\Omega_j\,\right]^2\left[\,\widetilde{\Delta}_j\,\right]^2}{16\left[\,1-\Omega_j\,\right]\left[\,\widetilde{\Delta}_j+\Delta_{\widetilde{P}j}\,\right]^2}\right\}\right). \tag{244}$$

(244) implies that:

$$\frac{\widetilde{\Theta}}{\Theta} \;<\; \frac{\left[\,4-\Omega_j\,\right]^2\left[\,\widetilde{\Delta}_j\,\right]^2}{16\left[\,1-\Omega_j\,\right]\left[\,\widetilde{\Delta}_j+\Delta_{\widetilde{P}j}\,\right]^2};\ \text{and} \tag{245}$$

$$\frac{\Theta}{\widetilde{\Theta}} \;<\; \frac{\left[\,4-\Omega_j\,\right]^2\left[\,\widetilde{\Delta}_j\,\right]^2}{16\left[\,1-\Omega_j\,\right]\left[\,\widetilde{\Delta}_j+\Delta_{Pj}\,\right]^2}. \tag{246}$$

(10), (238), (240), and (243) imply that for $k, i, j \in \{1,2\}$ and $k \neq i$:

$$\pi_{kj}^{C}\left(0\right) \;<\; \pi_{ij}^{N} \;\Leftrightarrow\; \frac{\Theta_k\left[\,\widetilde{\Delta}_{kj}+\Delta_{kj}\,\right]^2}{\beta_j^S\left[\,4-\Omega_j\,\right]^2} \;<\; \frac{\Theta_i\left[\,\widetilde{\Delta}_{ij}\,\right]^2}{16\,\beta_j^S\left[\,1-\Omega_j\,\right]} \;\Leftrightarrow\; \frac{\Theta_k}{\Theta_i} \;<\; \frac{\left[\,4-\Omega_j\,\right]^2\left[\,\widetilde{\Delta}_j\,\right]^2}{16\left[\,1-\Omega_j\,\right]\left[\,\widetilde{\Delta}_j+\Delta_{kj}\,\right]^2}. \tag{247}$$

[6] The superscript "N" in expression (238) denotes the setting in which Pk solely employs the no-entry instrument. Similarly, the superscript "C" in expression (240) denotes the setting in which Pk solely employs the commission instrument, and the superscript "B" in expression (241) denotes the setting in which Pk employs both instruments.

(247), $w_{kj} \geq 0$, and $\pi^C_{kj}(0) \geq \pi^C_{kj}(w_{kj})$ imply that:

$$\pi^C_{kj}(w_{kj}) \;<\; \pi^N_{ij} \;\Leftrightarrow\; \frac{\Theta_k}{\Theta_i} \;<\; \frac{[\,4-\Omega_j\,]^2 \left[\widetilde{\Delta}_j\right]^2}{16\,[\,1-\Omega_j\,]\left[\,\widetilde{\Delta}_j + \Delta_{kj}\,\right]^2}. \tag{248}$$

(245) and (248) imply that:

$$\pi^C_{\widetilde{P}j}\left(w_{\widetilde{P}j}\right) \;<\; \pi^N_{Pj}. \tag{249}$$

(246) and (248) imply that:

$$\pi^C_{Pj}\left(w_{Pj}\right) \;<\; \pi^N_{\widetilde{P}j}. \tag{250}$$

(249) implies that if $\widetilde{\text{P}}$ solely employs the commission instrument, then P can successfully attract both sellers by committing not to enter. (250) implies that if P solely employs the commission instrument, then $\widetilde{\text{P}}$ can successfully attract both sellers by committing not to enter. Consequently, no platform has an incentive to solely employ the commission instrument. Therefore, each platform either solely commits not to enter or employs both instruments in equilibrium under PC.

Suppose S1 sells on Pk and S2 sells on Pi $(k, i \in \{1, 2\})$ in equilibrium. Because both sellers face no competition from platforms, Lemma 1, (127), and (243) imply that consumer surplus under PC in this case is:

$$CS^C \;=\; \frac{\Theta_k}{2\,b^S_1}\left[\frac{\widetilde{\Delta}_1 - b^S_1\,w_{k1}}{2}\right]^2 + \frac{\Theta_i}{2\,b^S_2}\left[\frac{\widetilde{\Delta}_2 - b^S_2\,w_{i2}}{2}\right]^2, \tag{251}$$

where $w_{k1} \geq 0$ and $w_{i2} \geq 0$. (251), $w_{k1} \geq 0$, and $w_{i2} \geq 0$ imply that:

$$CS^C \;\leq\; \frac{\Theta_k}{2\,b^S_1}\left[\frac{\widetilde{\Delta}_1}{2}\right]^2 + \frac{\Theta_i}{2\,b^S_2}\left[\frac{\widetilde{\Delta}_2}{2}\right]^2. \tag{252}$$

Observe that:

$$\varsigma_{kj} \;\gtrless\; \frac{1}{2\,b^S_j}\left[\frac{\widetilde{\Delta}_{kj}}{2}\right]^2 \;\Leftrightarrow\; \varsigma_{kj} \;\gtrless\; \frac{1}{2\,\beta^S_j\,[\,1-\Omega_j\,]}\left[\frac{\widetilde{\Delta}_{kj}}{2}\right]^2$$

$$\Leftrightarrow\; \frac{1}{\beta^S_j\,\beta^P_j\,[\,1-\Omega_j\,]}\left\{\frac{\eta_j\,[\,2+\Omega_j\,]\,\Delta_{kj}}{8+\Omega_j}\left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]\right.$$
$$\left. + \frac{\beta^S_j}{2}\left[\frac{\overline{\Delta}_{kj}}{2} + \frac{\eta_j\,(\,2+\Omega_j\,)\,\Delta_{kj}}{2\beta^S_j\,(\,8+\Omega_j\,)}\right]^2 + \frac{\beta^P_j}{2}\left[\frac{(\,2+\Omega_j\,)\,\Delta_{kj}}{8+\Omega_j}\right]^2\right\}$$

$$\gtrless \frac{1}{8\,\beta_j^S\,[\,1-\Omega_j\,]}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}}{8+\Omega_j}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]+\frac{\beta_j^S}{2}\left[\frac{\overline{\Delta}_{kj}}{2}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{2\beta_j^S\,(\,8+\Omega_j\,)}\right]^2$$

$$+\frac{\beta_j^P}{2}\left[\frac{(\,2+\Omega_j\,)\;\Delta_{kj}}{8+\Omega_j}\right]^2 \gtrless \frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{2\,[\,8+\Omega_j\,]}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{2\,[\,8+\Omega_j\,]^2}$$

$$+\frac{\beta_j^S}{8}\left[\overline{\Delta}_{kj}+\frac{\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right]^2 \gtrless \frac{\beta_j^P}{8}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}$$

$$+\frac{\beta_j^S}{4}\left[\left(\overline{\Delta}_{kj}\right)^2+\frac{\left(\eta_j\right)^2(\,2+\Omega_j\,)^2\;(\Delta_{kj})^2}{\left(\beta_j^S\right)^2(\,8+\Omega_j\,)^2}+\frac{2\,\eta_j\,(\,2+\Omega_j\,)\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,(\,8+\Omega_j\,)}\right] \gtrless \frac{\beta_j^P}{4}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j}+\frac{\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{\beta_j^P\,[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2}+\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4}$$

$$+\frac{\beta_j^S\,\left[\eta_j\right]^2[\,2+\Omega_j\,]^2\;[\Delta_{kj}]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}+\frac{2\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]} \gtrless \frac{\beta_j^P}{4}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{4\,\beta_j^S\left[\eta_j\right]^2[\,2+\Omega_j\,]^2+4\left[\beta_j^S\right]^2\beta_j^P\,[\,2+\Omega_j\,]^2+\beta_j^S\left[\eta_j\right]^2[\,2+\Omega_j\,]^2}{4\left[\beta_j^S\right]^2[\,8+\Omega_j\,]^2}\,[\Delta_{kj}]^2$$

$$+\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4} \gtrless \frac{\beta_j^P}{4}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{4\left[\eta_j\right]^2+4\,\beta_j^S\,\beta_j^P+\left[\eta_j\right]^2}{4\,\beta_j^S\,[\,8+\Omega_j\,]^2}\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2$$

$$+\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{4\,\beta_j^S\,[\,8+\Omega_j\,]}+\frac{\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2}{4} \gtrless \frac{\beta_j^P}{4}\left[\Delta_{kj}+\frac{\eta_j}{\beta_j^P}\,\overline{\Delta}_{kj}\right]^2$$

$$\Leftrightarrow \frac{\left[5\left(\eta_j\right)^2+4\,\beta_j^S\,\beta_j^P\right][\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2}+\frac{6\,\beta_j^S\,\eta_j\,[\,2+\Omega_j\,]\;\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^S\,[\,8+\Omega_j\,]}+\beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2$$

$$\gtrless \ \beta_j^P \left[[\Delta_{kj}]^2 + \frac{(\eta_j)^2 \left(\overline{\Delta}_{kj}\right)^2}{\left(\beta_j^P\right)^2} + \frac{2\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}}{\beta_j^P} \right]$$

$$\Leftrightarrow \ \frac{\beta_j^S\,\beta_j^P\,[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{\beta_j^S\,[\,8+\Omega_j\,]^2} + \frac{6\,\eta_j\,[\,2+\Omega_j\,]\ \Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j} + \beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2$$

$$\gtrless \ \beta_j^P\,[\Delta_{kj}]^2 + \frac{\left[\eta_j\right]^2\,\left[\overline{\Delta}_{kj}\right]^2}{\beta_j^P} + 2\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$\Leftrightarrow \ \frac{\beta_j^P\,[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2\,[\Delta_{kj}]^2}{[\,8+\Omega_j\,]^2} - \beta_j^P\,[\Delta_{kj}]^2 + \frac{6\,\eta_j\,[\,2+\Omega_j\,]\ \Delta_{kj}\,\overline{\Delta}_{kj}}{8+\Omega_j} - 2\,\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+\ \beta_j^S\,\left[\overline{\Delta}_{kj}\right]^2 - \frac{\left[\eta_j\right]^2\,\left[\overline{\Delta}_{kj}\right]^2}{\beta_j^P} \ \gtrless \ 0$$

$$\Leftrightarrow \ \frac{[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2 - [\,8+\Omega_j\,]^2}{[\,8+\Omega_j\,]^2}\beta_j^P\,[\Delta_{kj}]^2 + \frac{6\,[\,2+\Omega_j\,] - 2\,[\,8+\Omega_j\,]}{8+\Omega_j}\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+\ \frac{\beta_j^P\,\beta_j^S - \left[\eta_j\right]^2}{\beta_j^P}\,\left[\overline{\Delta}_{kj}\right]^2 \ \gtrless \ 0$$

$$\Leftrightarrow \ \frac{[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2 - [\,8+\Omega_j\,]^2}{[\,8+\Omega_j\,]^2}\beta_j^P\,[\Delta_{kj}]^2 + \frac{4\,\Omega_j\ -4}{8+\Omega_j}\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+\ \frac{\beta_j^P\,\beta_j^S\,[\,1-\Omega_j\,]}{\beta_j^P}\,\left[\overline{\Delta}_{kj}\right]^2 \ \gtrless \ 0$$

$$\Leftrightarrow \ \frac{[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2 - [\,8+\Omega_j\,]^2}{[\,8+\Omega_j\,]^2}\beta_j^P\,[\Delta_{kj}]^2 - \frac{4\,[\,1-\Omega_j\,]}{8+\Omega_j}\eta_j\,\Delta_{kj}\,\overline{\Delta}_{kj}$$

$$+\ \beta_j^S\,[\,1-\Omega_j\,]\,\left[\overline{\Delta}_{kj}\right]^2 \ \gtrless \ 0$$

$$\Leftrightarrow \ \beta_j^S\,[\,1-\Omega_j\,]\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]^2 - \frac{4\,\eta_j\,[\,1-\Omega_j\,]}{8+\Omega_j}\,\frac{\overline{\Delta}_{kj}}{\Delta_{kj}} + \frac{\beta_j^P\,\left[\,(\,5\,\Omega_j+4\,)\,(\,2+\Omega_j\,)^2 - (\,8+\Omega_j\,)^2\,\right]}{[\,8+\Omega_j\,]^2} \ \gtrless \ 0 \tag{253}$$

Observe that:

$$[\,5\,\Omega_j+4\,]\,[\,2+\Omega_j\,]^2 - [\,8+\Omega_j\,]^2 \ = \ [\,5\,\Omega_j+4\,]\left[\,4+(\Omega_j)^2+4\,\Omega_j\,\right] - \left[\,64+[\Omega_j]^2+16\,\Omega_j\,\right]$$

$$= \ 5\,\Omega_j\left[\,4+(\Omega_j)^2+4\,\Omega_j\,\right] + 4\left[\,4+(\Omega_j)^2+4\,\Omega_j\,\right] - 64 - [\Omega_j]^2 - 16\,\Omega_j$$

$$= \ 20\,\Omega_j + 5\,[\Omega_j]^3 + 20\,[\Omega_j]^2 + 16 + 4\,[\Omega_j]^2 + 16\,\Omega_j - 64 - [\Omega_j]^2 - 16\,\Omega_j$$

$$= \ 20\,\Omega_j + 5\,[\Omega_j]^3 + 23\,[\Omega_j]^2 - 48 \ < \ 0. \tag{254}$$

(20), (253), and (254) imply that:

$$\varsigma_{kj} \gtrless \frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_{kj}}{2}\right]^2 \;\;\Leftrightarrow\;\; \overline{a}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]^2 + \overline{b}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right] + \overline{c} \gtrless 0, \tag{255}$$

$$\overline{a} > 0,\, \overline{b} < 0, \text{ and } \overline{c} < 0. \tag{256}$$

(256) implies that:

$$\overline{a}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]^2 + \overline{b}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right] + \overline{c} > 0 \;\;\Leftrightarrow\;\; \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > \frac{-\overline{b} + \sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}; \tag{257}$$

$$\overline{a}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right]^2 + \overline{b}\left[\frac{\overline{\Delta}_{kj}}{\Delta_{kj}}\right] + \overline{c} < 0 \;\;\Leftrightarrow\;\; \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} \in \left(0, \frac{-\overline{b} + \sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}\right). \tag{258}$$

(255), (257), and (243) imply that:

$$\varsigma_{kj} > \frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 \;\;\Leftrightarrow\;\; \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} > \frac{-\overline{b} + \sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}. \tag{259}$$

(255), (258), and (243) imply that:

$$\varsigma_{kj} < \frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 \;\;\Leftrightarrow\;\; \frac{\overline{\Delta}_{kj}}{\Delta_{kj}} \in \left(0, \frac{-\overline{b} + \sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}\right). \tag{260}$$

Case I. Both sellers sell on P in equilibrium under PC.

(251) and (252) imply that:

$$CS^C = \frac{\Theta}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_1 - b_1^S\,w_{P1}}{2}\right]^2 + \frac{\Theta}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_2 - b_2^S\,w_{P2}}{2}\right]^2 \leq \frac{\Theta}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_1}{2}\right]^2 + \frac{\Theta}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_2}{2}\right]^2. \tag{261}$$

(259) implies that if P's relative selling strength is sufficiently pronounced (i.e., $\frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} > \frac{-\overline{b}+\sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}}$), then

$$\frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 < \varsigma_{Pj}. \tag{262}$$

(262) implies that if $\frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} > \frac{-\bar{b}+\sqrt{\left(\bar{b}\right)^2 - 4\,\bar{a}\,\bar{c}}}{2\,\bar{a}}$:

$$\frac{\Theta}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_1}{2}\right]^2 + \frac{\Theta}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_2}{2}\right]^2 \;<\; \Theta\,\varsigma_{P1} + \Theta\,\varsigma_{P2}. \tag{263}$$

(237), (261), and (263) imply that:

$$CS^C \;<\; CS^M \;\;\text{if}\;\; \frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} \;>\; \frac{-\bar{b}+\sqrt{\left(\bar{b}\right)^2 - 4\,\bar{a}\,\bar{c}}}{2\,\bar{a}}. \tag{264}$$

<u>Case II</u>. Both sellers sell on $\widetilde{\text{P}}$ in equilibrium under PC.

(251) and (252) imply that:

$$CS^C \;=\; \frac{\widetilde{\Theta}}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_1 - b_1^S\,w_{\widetilde{P}1}}{2}\right]^2 + \frac{\widetilde{\Theta}}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_2 - b_2^S\,w_{\widetilde{P}2}}{2}\right]^2 \;\leq\; \frac{\widetilde{\Theta}}{2\,b_1^S}\left[\frac{\widetilde{\Delta}_1}{2}\right]^2 + \frac{\widetilde{\Theta}}{2\,b_2^S}\left[\frac{\widetilde{\Delta}_2}{2}\right]^2. \tag{265}$$

Observe that:

$$\frac{\widetilde{\Theta}}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 \;<\; \Theta\,\varsigma_{Pj} \;\;\Leftrightarrow\;\; \frac{\widetilde{\Theta}}{\Theta} \;<\; \frac{\varsigma_{Pj}}{\frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2}. \tag{266}$$

(237), (265), and (266) imply that:

$$CS^C < CS^M \;\;\text{if}\;\; \frac{\widetilde{\Theta}}{\Theta} \;<\; \frac{\varsigma_{Pj}}{\frac{1}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2}. \tag{267}$$

<u>Case III</u>. Sj sells on P and Sl sells on $\widetilde{\text{P}}$ in equilibrium under PC ($j, l \in \{1, 2\}$, $j \neq l$).

(251) and (252) imply that:

$$CS^C \;=\; \frac{\Theta}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j - b_j^S\,w_{Pj}}{2}\right]^2 + \frac{\widetilde{\Theta}}{2\,b_l^S}\left[\frac{\widetilde{\Delta}_l - b_l^S\,w_{\widetilde{P}l}}{2}\right]^2 \;\leq\; \frac{\Theta}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 + \frac{\widetilde{\Theta}}{2\,b_l^S}\left[\frac{\widetilde{\Delta}_l}{2}\right]^2. \tag{268}$$

(262) implies that if $\frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} > \frac{-\bar{b}+\sqrt{\left(\bar{b}\right)^2 - 4\,\bar{a}\,\bar{c}}}{2\,\bar{a}}$:

$$\frac{\Theta}{2\,b_j^S}\left[\frac{\widetilde{\Delta}_j}{2}\right]^2 \;<\; \Theta\,\varsigma_{Pj}. \tag{269}$$

(266) implies that if $\frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{Pl}}{\frac{1}{2\,b_l^S}\left[\frac{\widetilde{\Delta}_l}{2}\right]^2}$:

$$\frac{\widetilde{\Theta}}{2\,b_l^S}\left[\frac{\widetilde{\Delta}_l}{2}\right]^2 \; < \; \Theta\,\varsigma_{Pl}. \tag{270}$$

(237), (268) - (270) imply that:

$$CS^C \; < \; CS^M \;\; \text{if} \;\; \frac{\overline{\Delta}_{Pj}}{\Delta_{Pj}} > \frac{-\overline{b} + \sqrt{\left(\overline{b}\right)^2 - 4\,\overline{a}\,\overline{c}}}{2\,\overline{a}} \;\; \text{and} \;\; \frac{\widetilde{\Theta}}{\Theta} < \frac{\varsigma_{Pl}}{\frac{1}{2\,b_l^S}\left[\frac{\widetilde{\Delta}_l}{2}\right]^2}. \;\; \blacksquare \tag{271}$$

II. Numerical solutions.

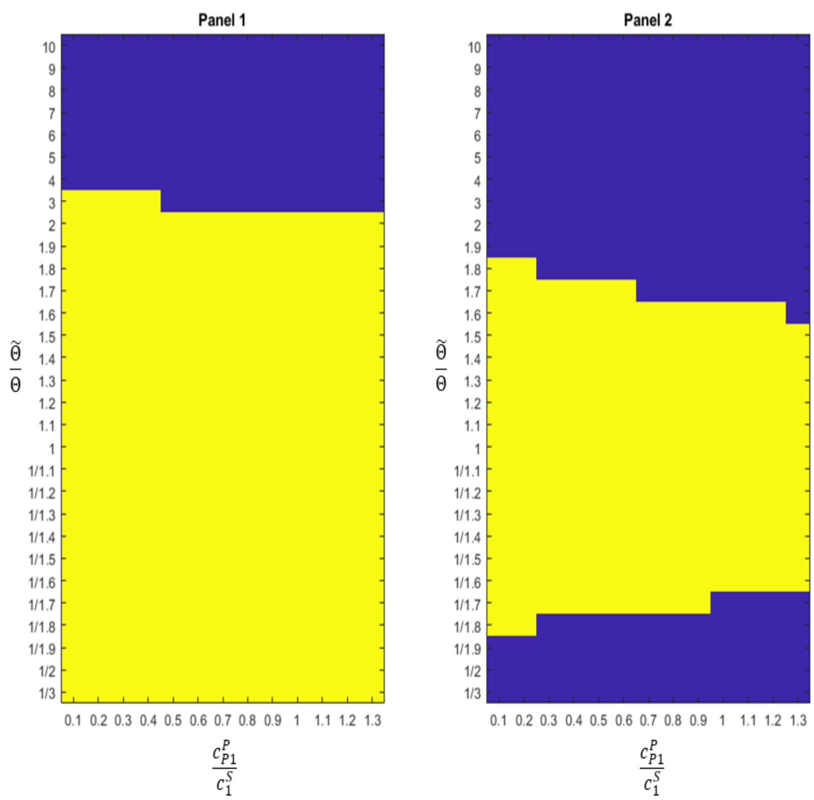

Figure 1: Comparison between CS and CS^M when $\theta_j = 1$.

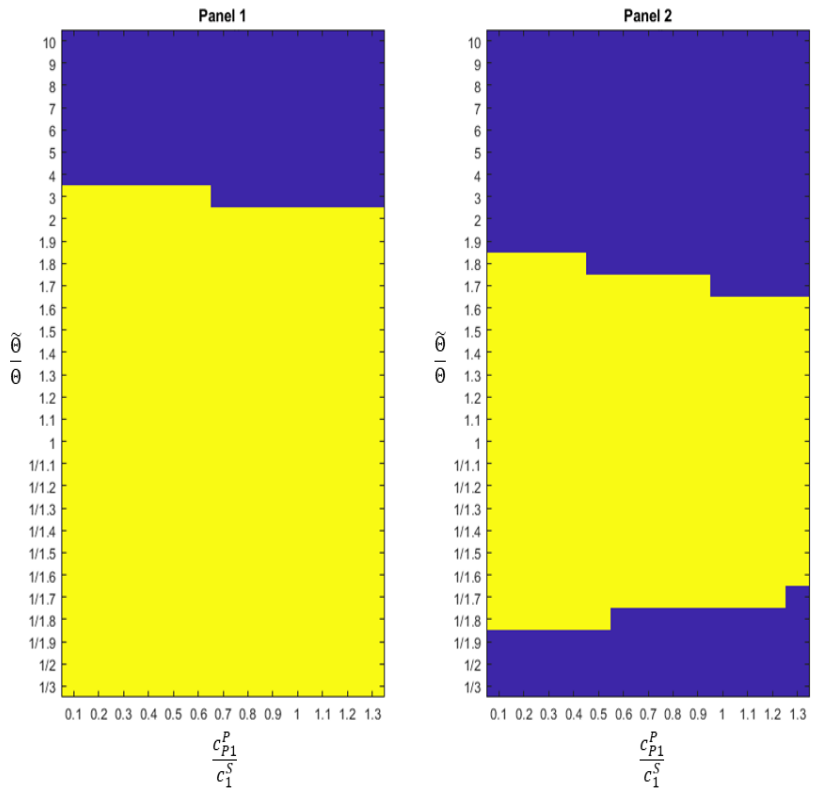

Figure 2: Comparison between CS and CS^M when $\theta_j = 1.1$.

III. $\widetilde{\mathrm{P}}$ is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$).

In the case where $\widetilde{\mathrm{P}}$ is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$), increased platform competition continues to reduce seller competition when platforms have comparable platform strengths. However, increased platform competition can increase seller competition when $\widetilde{\mathrm{P}}$ is a sufficiently stronger platform than P. Proposition 10 states these results formally. Therefore, my key qualitative conclusion – increased platform competition often harms consumers – seem likely to persist in the case where platforms have comparable platform strengths and $\frac{\widetilde{c}_j^P}{c_j^P} < 1$ in the sense that increased platform competition continues to reduce seller competition. However, increased platform competition that induces both sellers to sell on $\widetilde{\mathrm{P}}$ that is sufficiently stronger than P (i.e., $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$ and $\frac{\widetilde{c}_j^P}{c_j^P} < 1$) might benefit consumers by promoting seller competition.

Proposition 10. *Suppose Condition FS and Assumption BC hold. Further suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$). Then increased platform competition: (i) reduces platform-seller competition if $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \phi_1\right)$; and (ii) increases platform-seller competition if $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.*

Proof. Proposition 1 implies that Sj ($j \in \{1, 2\}$) competes against P under MP.

First suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a stronger platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} > 1$) under PC, where $\widetilde{\Theta}$ denotes $\widetilde{\mathrm{P}}$'s platform strength.

Case I. $\frac{\widetilde{\Theta}}{\Theta} > \phi_2$.

Proposition 2 implies that S1 and S2 competes against $\widetilde{\mathrm{P}}$ under PC. Because $\widetilde{\mathrm{P}}$ is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$), increased platform competition increases platform-seller competition in the sense that each seller faces increased competition under PC, whereas each seller competes against P under MP.[7]

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in (\phi_1, \phi_2)$.

Proposition 2 implies that S1 competes against $\widetilde{\mathrm{P}}$ whereas S2 faces no competition under PC. Because $\widetilde{\mathrm{P}}$ is a stronger seller than P (i.e., $\frac{\widetilde{c}_j^P}{c_j^P} < 1$), the effect of increased platform competition on platform-seller competition is ambiguous because S1 faces increased competition and S2 faces no competition under PC, whereas each seller competes against P under MP.

Case III. $\frac{\widetilde{\Theta}}{\Theta} \in (1, \phi_1)$.

[7] $\widetilde{c}_j^P$ denotes $\widetilde{\mathrm{P}}$'s cost of imitating Sj's product, and c_j^P denotes P's cost of imitating Sj's product.

Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Next suppose P faces a symmetric competing platform $\widetilde{\mathrm{P}}$ (i.e., $\frac{\widetilde{\Theta}}{\Theta} = 1$) under PC. Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Finally, suppose P faces a competing platform $\widetilde{\mathrm{P}}$ that is a weaker platform (i.e., $\frac{\widetilde{\Theta}}{\Theta} < 1$) under PC.

Case I. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_1}, 1 \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} < \phi_1$. Proposition 2 implies that each seller faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that each seller faces no competition under PC, whereas each seller competes against P under MP.

Case II. $\frac{\widetilde{\Theta}}{\Theta} \in \left(\frac{1}{\phi_2}, \frac{1}{\phi_1} \right)$.

In this case, $\frac{\Theta}{\widetilde{\Theta}} \in (\phi_1, \phi_2)$. Proposition 2 implies that S1 competes against P whereas S2 faces no competition under PC. Therefore, increased platform competition reduces platform-seller competition in the sense that S2 faces no competition and S1 faces unchanged competition under PC, whereas each seller competes against P under MP. ■